# MMAT5390: Mathematical Image Processing

## Solutions to Chapter 4 Exercises

1. $H_{\text{i}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, $H_{\text{ii}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$$H_{\text{iii}} = \begin{pmatrix} 1 & \frac{16}{17} & \frac{1}{2} & \frac{16}{17} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \\ \frac{1}{2} & \frac{16}{41} & \frac{1}{5} & \frac{16}{41} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \end{pmatrix}, \; H_{\text{iv}} = \begin{pmatrix} 0 & \frac{1}{17} & \frac{1}{2} & \frac{1}{17} \\ \frac{1}{17} & \frac{1}{5} & \frac{25}{41} & \frac{1}{5} \\ \frac{1}{2} & \frac{25}{41} & \frac{4}{5} & \frac{25}{41} \\ \frac{1}{17} & \frac{1}{5} & \frac{25}{41} & \frac{1}{5} \end{pmatrix},$$

$$H_{\text{v}} = \begin{pmatrix} 1 & e^{-\frac{1}{8}} & e^{-\frac{1}{2}} & e^{-\frac{1}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \\ e^{-\frac{1}{2}} & e^{-\frac{5}{8}} & e^{-1} & e^{-\frac{5}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \end{pmatrix}, \; H_{\text{vi}} = \begin{pmatrix} 0 & 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{2}} & 1-e^{-\frac{1}{8}} \\ 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{4}} & 1-e^{-\frac{5}{8}} & 1-e^{-\frac{1}{4}} \\ 1-e^{-\frac{1}{2}} & 1-e^{-\frac{5}{8}} & 1-e^{-1} & 1-e^{-\frac{5}{8}} \\ 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{4}} & 1-e^{-\frac{5}{8}} & 1-e^{-\frac{1}{4}} \end{pmatrix}.$$

(a)

$$DFT(f_1) = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$
$$= \frac{1}{16}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

i,iii,v.

$$\tilde{f}_1 = Re(iDFT(DFT(f_1))) = f_1.$$

ii,iv,vi.

$$\tilde{f}_1 = Re(iDFT(\mathbf{0})) = \mathbf{0}.$$

(b)

$$DFT(f_2) = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$
$$= \frac{1}{16}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

i.

$$\tilde{f}_2 = Re(iDFT(\mathbf{0})) = \mathbf{0}.$$

ii.

$$\tilde{f}_2 = Re(iDFT(DFT(f_2))) = f_2.$$

iii.

$$\tilde{f}_2 = Re(iDFT(\frac{1}{5}DFT(f_2))) = \frac{1}{5}f_2.$$

iv.

$$\tilde{f}_2 = Re(iDFT(\frac{4}{5}DFT(f_2))) = \frac{4}{5}f_2.$$

v.

$$\tilde{f_2} = Re(iDFT(e^{-1}DFT(f_2))) = e^{-1}f_2.$$

vi.

$$\tilde{f_2} = Re(iDFT((1-e^{-1})DFT(f_2))) = (1-e^{-1})f_2.$$

(c)

$$\begin{aligned}
DFT(f_3) &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}0 & 2 & 2 & 0\\ 0 & -1-j & -1-j & 0\\ 0 & 0 & 0 & 0\\ 0 & -1+j & -1+j & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\\
&= \frac{1}{16}\begin{pmatrix}4 & -2-2j & 0 & -2+2j\\ -2-2j & 2j & 0 & 2\\ 0 & 0 & 0 & 0\\ -2+2j & 2 & 0 & -2j\end{pmatrix}.
\end{aligned}$$

i.

$$\tilde{f_3} = iDFT(DFT(f_3)) = f_3.$$

ii.

$$\tilde{f_3} = iDFT(\mathbf{0}) = \mathbf{0}.$$

iii.

$$\begin{aligned}
\tilde{f_3} &= \frac{1}{16}Re(iDFT\left(\begin{pmatrix}4 & -\frac{32}{17}(1+j) & 0 & \frac{32}{17}(-1+j)\\ -\frac{32}{17}(1+j) & \frac{8}{5}j & 0 & \frac{8}{5}\\ 0 & 0 & 0 & 0\\ \frac{32}{17}(-1+j) & \frac{8}{5} & 0 & -\frac{8}{5}j\end{pmatrix}\right))\\
&= \frac{1}{16}Re(\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\begin{pmatrix}4 & -\frac{32}{17}(1+j) & 0 & \frac{32}{17}(-1+j)\\ -\frac{32}{17}(1+j) & \frac{8}{5}j & 0 & \frac{8}{5}\\ 0 & 0 & 0 & 0\\ \frac{32}{17}(-1+j) & \frac{8}{5} & 0 & -\frac{8}{5}j\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\\
&= \frac{1}{340}\begin{pmatrix}-7 & 17 & 17 & -7\\ 17 & 313 & 313 & 17\\ 17 & 313 & 313 & 17\\ -7 & 17 & 17 & -7\end{pmatrix}.
\end{aligned}$$

iv.

$$\begin{aligned}
\tilde{f_3} &= \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} - \frac{1}{340}\begin{pmatrix}-7 & 17 & 17 & -7\\ 17 & 313 & 313 & 17\\ 17 & 313 & 313 & 17\\ -7 & 17 & 17 & -7\end{pmatrix}\\
&= \frac{1}{340}\begin{pmatrix}7 & -17 & -17 & 7\\ -17 & 27 & 27 & -17\\ -17 & 27 & 27 & -17\\ 7 & -17 & -17 & 7\end{pmatrix}.
\end{aligned}$$

v.

$$\begin{aligned}
\tilde{f_3} &= \frac{1}{16}Re(iDFT\left(\begin{pmatrix}4 & -2e^{-\frac{1}{8}}(1+j) & 0 & 2e^{-\frac{1}{8}}(-1+j)\\ -2e^{-\frac{1}{8}}(1+j) & 2e^{-\frac{1}{4}}j & 0 & 2e^{-\frac{1}{4}}\\ 0 & 0 & 0 & 0\\ 2e^{-\frac{1}{8}}(-1+j) & 2e^{-\frac{1}{4}} & 0 & -2e^{-\frac{1}{4}}j\end{pmatrix}\right))\\
&\approx \begin{pmatrix}0.0035 & 0.0553 & 0.0553 & 0.0035\\ 0.0553 & 0.8859 & 0.8859 & 0.0553\\ 0.0553 & 0.8859 & 0.8859 & 0.0553\\ 0.0035 & 0.0553 & 0.0553 & 0.0035\end{pmatrix}.
\end{aligned}$$

vi.

$$\tilde{f}_3 \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0.0035 & 0.0553 & 0.0553 & 0.0035 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0035 & 0.0553 & 0.0553 & 0.0035 \end{pmatrix}$$
$$= \begin{pmatrix} -0.0035 & -0.0553 & -0.0553 & -0.0035 \\ -0.0553 & 0.1141 & 0.1141 & -0.0553 \\ -0.0553 & 0.1141 & 0.1141 & -0.0553 \\ -0.0035 & -0.0553 & -0.0553 & -0.0035 \end{pmatrix}$$

2. (a) Since $M \geq 6$ and $N \geq 8$, $D(1,2) = 5$, $D(3,1) = 10$ and $D(2,4) = 20$.
Since $3H_1(1,2) = 4H_1(3,1) = 6H_1(2,4)$,

$$\frac{3D_0^{2n}}{D_0^{2n} + 5^n} = \frac{4D_0^{2n}}{D_0^{2n} + 10^n} = \frac{6D_0^{2n}}{D_0^{2n} + 20^n}.$$

Since $D_0 > 0$,

$$\begin{cases} 3(D_0^{2n} + 10^n) & = 4(D_0^{2n} + 5^n), \\ 4(D_0^{2n} + 20^n) & = 6(D_0^{2n} + 10^n), \end{cases}$$

which gives

$$\begin{cases} D_0^{2n} & = 3 \cdot 10^n - 4 \cdot 5^n = 5^n(3 \cdot 2^n - 4), \\ 2D_0^{2n} & = 4 \cdot 20^n - 6 \cdot 10^n \implies D_0^{2n} = 2 \cdot 20^n - 3 \cdot 10^n = 5^n(2 \cdot 2^{2n} - 3 \cdot 2^n). \end{cases}$$

Hence $3 \cdot 2^n - 4 = 2 \cdot 2^{2n} - 3 \cdot 2^n$, and thus $(2^n - 1)(2^n - 2) = 0$ and $n = 0$ or $1$.
If $n = 0$, then $1 = D_0^{2\times 0} = 5^0(3 \cdot 2^0 - 4) = -1$. Contradiction.
Hence $n = 1$. Since $D_0^2 = 5(3 \cdot 2 - 4)$, $D_0 = \sqrt{10}$.

(b) Since $M \geq 2c > 2a$ and $N \geq 4c$, $D(a,b) = a^2 + b^2 = c^2$ and $D(0,2c) = 4c^2$.
Given $H_2(a,b) = 8H_2(0,2c)$, i.e. $e^{-\frac{c^2}{2\sigma^2}} = 8e^{-\frac{2c^2}{\sigma^2}}$.
Hence $e^{-\frac{c^2}{2\sigma^2}}(2e^{-\frac{c^2}{2\sigma^2}} - 1)(4e^{-\frac{c^2}{\sigma^2}} + 2e^{-\frac{c^2}{2\sigma^2}} + 1) = 0$.
Thus $e^{-\frac{c^2}{2\sigma^2}} = \frac{1}{2}$; $-\frac{c^2}{2\sigma^2} = -\ln 2$ and thus $\sigma = \frac{c}{\sqrt{2\ln 2}}$.

(c) Since $M = N = 10$, $D(2,7) = 2^2 + 3^2 = 13$ and $D(5,1) = 5^2 + 1^2 = 26$.
Given $H_3(2,7) = \frac{1}{3}$ and $H_3(5,1) = \frac{4}{5}$, i.e.

$$\begin{cases} \frac{13^n}{D_0^{2n} + 13^n} & = \frac{1}{3}, \\ \frac{26^n}{D_0^{2n} + 26^n} & = \frac{4}{5}. \end{cases}$$

Then $D_0^{2n} = 2 \cdot 13^n$ and $4D_0^{2n} = 26^n$, and thus $4 = 2^{n-1}$.
Hence $n = 3$; then $D_0^6 = D_0^{2n} = 2 \cdot 13^n = 2 \cdot 13^3$, and thus $D_0 = \sqrt[6]{2}\sqrt{13}$.

3. Recall that for any $f \in M_{M\times N}(\mathbb{R})$, $DFT(h * f)(u,v) = MN\,DFT(h)(u,v)DFT(f)(u,v)$; hence $H(u,v) = MN\,DFT(h)(u,v)$.

(a)

$$H_1(u,v) = \sum_{x=-k}^{k}\sum_{y=-k}^{k} \frac{1}{(2k+1)^2} e^{-2\pi j(\frac{ux}{M} + \frac{vy}{N})} = \frac{1}{(2k+1)^2} \sum_{x=-k}^{k} e^{-2\pi j\frac{ux}{M}} \sum_{y=-k}^{k} e^{-2\pi j\frac{vy}{N}}$$
$$= \frac{1}{(2k+1)^2}[1 + 2\sum_{x=1}^{k} \cos\frac{2\pi ux}{M}][1 + 2\sum_{y=1}^{k}\cos\frac{2\pi vy}{N}].$$

(b) $H_2(u,v) = \frac{r}{r+4} + \frac{1}{r+4}(e^{-2\pi j\frac{u}{M}} + e^{2\pi j\frac{u}{M}} + e^{-2\pi j\frac{v}{N}} + e^{2\pi j\frac{v}{N}}) = \frac{r + 2(\cos\frac{2\pi u}{M} + \cos\frac{2\pi v}{N})}{r+4}$.

(c)

$$\begin{aligned}
H_3(u,v) &= \frac{1}{4} + \frac{1}{8}(e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}}) \\
&\quad + \frac{1}{16}(e^{-2\pi j(\frac{u}{M}+\frac{v}{N})} + e^{-2\pi j(\frac{u}{M}-\frac{v}{N})} + e^{-2\pi j(-\frac{u}{M}+\frac{v}{N})} + e^{-2\pi j(-\frac{u}{M}-\frac{v}{N})}) \\
&= \frac{1}{4} + \frac{1}{4}(\cos\frac{2\pi u}{M} + \cos\frac{2\pi v}{N}) + \frac{1}{4}\cos\frac{2\pi u}{M}\cos\frac{2\pi v}{N} \\
&= \frac{1}{4}(\cos\frac{2\pi u}{M} + 1)(\cos\frac{2\pi v}{N} + 1) \\
&= \cos^2\frac{\pi u}{M}\cos^2\frac{\pi v}{N}.
\end{aligned}$$

(d)

$$\begin{aligned}
H_4(u,v) &= -4 + e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}} \\
&= -4 + 2\cos\frac{2\pi u}{M} + 2\cos\frac{2\pi v}{N} \\
&= -4(\sin^2\frac{\pi u}{M} + \sin^2\frac{\pi v}{N}).
\end{aligned}$$

(e)

$$\begin{aligned}
H_5(u,v) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{-2\pi j(\frac{atu}{M}+\frac{btv}{N})} \\
&= \begin{cases} \frac{1}{T}\cdot\frac{1-e^{-2\pi jT(\frac{au}{M}+\frac{bv}{N})}}{1-e^{-2\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{e^{\pi jT(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi jT(\frac{au}{M}+\frac{bv}{N})}}{e^{\pi j(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{\sin(\pi T(\frac{au}{M}+\frac{bv}{N}))}{\sin(\pi(\frac{au}{M}+\frac{bv}{N}))} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

4. To be supplemented.

5.

$$\begin{aligned}
&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} F(m,n)\overline{F(m,n)} \\
&= \frac{1}{M^2N^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\overline{f(k',l')}e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})} \\
&= \frac{1}{M^2N^2}\sum_{m,k,k'=0}^{M-1}\sum_{n,l,l'=0}^{N-1} f(k,l)\overline{f(k',l')}e^{2\pi j(\frac{m(k'-k)}{M}+\frac{n(l'-l)}{N})} \\
&= \frac{1}{M^2N^2}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} f(k,l)\overline{f(k',l')}\cdot M\mathbf{1}_{M\mathbb{Z}}(k'-k)\cdot N\mathbf{1}_{N\mathbb{Z}}(l'-l) \\
&= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}|f(k,l)|^2.
\end{aligned}$$

6. (a) Let $f$ be a minimizer of $\|L\vec{f}\|_F^2$ over all $\{\varphi \in M_{M\times N}(\mathbb{R}) : \|\vec{g} - D\vec{f}\|_2^2 = \varepsilon\}$. Then the method of Lagrange multipliers states that $f$ necessarily minimizes the following Lagrangian:

$$\mathcal{L}(f) = \|L\vec{f}\|_2^2 + \lambda\|\vec{g} - D\vec{f}\|_2^2.$$

Hence $\frac{d\mathcal{L}}{d\vec{f}} = 0$, i.e.

$$\frac{d}{d\vec{f}}[\vec{f}^T L^T L f + \lambda(\vec{g}^T - \vec{f}^T D^T)(\vec{g} - Df)] = 0,$$

which gives $2L^T L\vec{f} + \lambda(-2D^T\vec{g} + 2D^T D\vec{f}) = 0$ and thus $(\lambda D^T D + L^T L)\vec{f} = \lambda D^T\vec{g}$.

(b) i. For any $x, y \in \{0, 1, \cdots, M-1\}$,

$$(\sqrt{M}U_M)\overline{(\sqrt{M}U_M)}(x,y) = M\sum_{k=0}^{M-1} U_M(x,k)\overline{U_M(k,y)} = \frac{1}{M}\sum_{k=0}^{M-1} e^{2\pi j\frac{k(y-x)}{M}}$$
$$= \frac{1}{M}\cdot M\delta(y-x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\sqrt{M}U_M$ is unitary.

ii. Note that the $(k,l)$-th block of $U \otimes V$, denoted here by $(U \otimes V)_{k,l}$, is given by $U(k,l)V$. Hence the $(k,l)$-th block of $(U \otimes V)^*$ is given by

$$[(U \otimes V)^*]_{k,l} = [(U \otimes V)_{l,k}]^* = \overline{U(l,k)}V^* = U^*(k,l)V^*.$$

Hence for any $k, l \in \{0, 1, \cdots, N\}$,

$$\begin{aligned} [(U \otimes V)(U \otimes V)^*]_{k,l} &= \sum_{n=0}^{N-1}(U \otimes V)_{k,n}[(U \otimes V)^*]_{n,l} \\ &= \sum_{n=0}^{N-1} U(k,n)U^*(n,l)VV^* \\ &= \delta(k-l)I_M. \end{aligned}$$

Hence $U \otimes V$ is unitary.

iii. For any $k, l \in \{0, 1, \cdots, M\}$,

$$\begin{aligned} U_M C\overline{U_M}(k,l) &= \sum_{m=0}^{M-1} U_M(k,m)(C\overline{U_M})(m,l) \\ &= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} U_M(k,m)c_{m-n}\overline{U_M}(n,l) \\ &= \frac{1}{M^2}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} c_{m-n}e^{2\pi j\frac{ln-mk}{M}} \\ &= \frac{1}{M^2}\sum_{m=0}^{M-1}\sum_{n'=m-M+1}^{m} c_{n'}e^{2\pi j\frac{l(m-n')-mk}{M}} \\ &= \frac{1}{M^2}\sum_{n'=0}^{M-1} c_{n'}e^{-2\pi j\frac{ln'}{M}}\sum_{m=0}^{M-1} e^{2\pi j\frac{m(l-k)}{M}} \\ &= \frac{1}{M}\sum_{n=0}^{M-1} c_n e^{-2\pi j\frac{ln}{M}}\delta(l-k). \end{aligned}$$

Hence $U_M C\overline{U_M}$ is diagonal.
Since $\sqrt{M}U_M$ is unitary, $U_M^{-1} = MU_M^* = M\overline{U_M}$, and thus $MU_M C\overline{U_M}$ is a diagonalization of $C$. Hence the eigenvalues of $C$ are given by the diagonal entries of $MU_M C\overline{U_M}$:

$$\lambda_l(C) = \sum_{n=0}^{M-1} c_n e^{-2\pi j\frac{ln}{M}},\ l = 0, 1, \cdots, M-1.$$

iv. For any $k, l \in \{0, 1, \cdots, N-1\}$, the $(k, l)$-th block of $(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$, denoted here by $[(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})]_{k,l}$, is given by:

$$\begin{aligned}
[(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})]_{k,l} &= \sum_{m=0}^{N-1} (U_N \otimes U_M)_{k,m} [D(\overline{U}_N \otimes \overline{U_M})]_{m,l} \\
&= \sum_{m=0}^{N-1} (U_N \otimes U_M)_{k,m} \sum_{n=0}^{N-1} D_{m-n} (\overline{U_N} \otimes \overline{U_M})_{n,l} \\
&= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} U_N(k,m) \overline{U_N(n,l)} U_M D_{m-n} \overline{U_M} \\
&= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} U_M D_{m-n} \overline{U_M} \\
&= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n'=m-N+1}^{m} e^{2\pi j \frac{l(m-n')-km}{N}} U_M D_{n'} \overline{U_M} \\
&= \frac{1}{N^2} \sum_{m=0}^{N-1} e^{2\pi j \frac{m(l-k)}{N}} \sum_{n'=0}^{N-1} e^{-2\pi j \frac{ln'}{N}} U_M D_{n'} \overline{U_M} \\
&= \frac{1}{N} \delta(l-k) \sum_{n'=0}^{N-1} e^{-2\pi j \frac{ln'}{N}} U_M D_{n'} \overline{U_M}.
\end{aligned}$$

Since $D$ is block-circulant, $D_{n'}$ is circulant. Hence by the result of (b)iii., $U_M D_{n'} \overline{U_M}$ is diagonal and thus $(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$ is diagonal.
Since $\sqrt{N}U_N$ and $\sqrt{M}U_M$ are unitary, $\sqrt{MN}U_N \otimes U_M$ is unitary, and $MN(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$ is a diagonalization of $D$. The eigenvalues of $D$ are the diagonal entries of $MN(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$, i.e.

$$\lambda_i(D) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} D_{n,m} e^{-2\pi j (\frac{km}{M} + \frac{ln}{N})},\ k \in \{0, 1, \cdots, M-1\},\ l \in \{0, 1, \cdots, N-1\},$$

where $D_{n,m}$ denotes the value of the entries on the diagonal of $D_n$ with indices $\{(x, y) : x - y \in m + M\mathbb{Z}\}$.

(c) i. For any $x, y \in \{0, 1, \cdots, M-1\}$,

$$\begin{aligned}
W_M D_{k-l} \overline{W_M}(x,y) &= \sum_{s=0}^{M-1} W_M(x,s) [D_{k-l} \overline{W_M}](s,y) \\
&= \sum_{s=0}^{M-1} W_M(x,s) \sum_{t=0}^{M-1} D_{k-l}(s,t) \overline{W_M}(t,y) \\
&= \frac{1}{M} \sum_{s=0}^{M-1} \sum_{t=0}^{M-1} D_{k-l,s-t} e^{2\pi j \frac{sx-ty}{M}} \\
&= \frac{1}{M} \sum_{s=0}^{M-1} \sum_{t'=s-M+1}^{s} D_{k-l,t'} e^{2\pi j \frac{sx-(s-t')y}{M}} \\
&= \frac{1}{M} \sum_{s=0}^{M-1} e^{2\pi j \frac{s(x-y)}{M}} \sum_{t'=0}^{M-1} D_{k-l,t'} e^{2\pi j \frac{t'y}{M}} \\
&= \begin{cases} \sum\limits_{m=0}^{M-1} D_{k-l,m} e^{2\pi j \frac{my}{M}} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

ii. For any $k, l \in \mathbb{Z} \cap [0, N-1]$,

$$\begin{aligned}
[(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})]_{k,l} &= \sum_{m=0}^{N-1} (W_N \otimes W_M)_{k,m} [D(\overline{W_N} \otimes \overline{W_M})]_{m,l} \\
&= \sum_{m=0}^{N-1} (W_N \otimes W_M)_{k,m} \sum_{n=0}^{N-1} D_{m-n} (\overline{W_N} \otimes \overline{W_M})_{n,l} \\
&= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} W_N(k,m) \overline{W_N(n,l)} W_M D_{m-n} \overline{W_M} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi j \frac{km - ln}{N}} W_M D_{m-n} \overline{W_M} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n'=0}^{N-1} e^{2\pi j \frac{km - l(m-n')}{N}} W_M D_{n'} \overline{W_M} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi j \frac{m(k-l)}{N}} \sum_{n'=0}^{N-1} e^{2\pi j \frac{ln'}{N}} W_M D_{n'} \overline{W_M} \\
&= \delta(k-l) \sum_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}} W_M D_n \overline{W_M}.
\end{aligned}$$

Hence for any $k, l \in \{0, 1, \cdots, N-1\}$ and $x, y \in \{0, 1, \cdots, M-1\}$,

$$\begin{aligned}
& [(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})](x + kM, y + lM) \\
&= [(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})]_{k,l}(x, y) \\
&= \begin{cases} \sum\limits_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}} [W_M D_n \overline{W_M}](x, y) & \text{if } k = l, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum\limits_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}} \sum\limits_{m=0}^{M-1} D_{n,m} e^{2\pi j \frac{mx}{M}} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum\limits_{m=0}^{M-1} \sum\limits_{n=0}^{N-1} D(m + nM, 0) e^{2\pi j (\frac{mx}{M} + \frac{ln}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sum\limits_{m=0}^{M-1} \sum\limits_{n=0}^{N-1} h(m, n) e^{2\pi j (\frac{mx}{M} + \frac{kn}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} MN\, \overline{DFT(h)(x, k)} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(d) For any $x \in \{0, 1, \cdots, M-1\}$ and $y \in \{0, 1, \cdots, N-1\}$, and $f \in M_{M \times N}(\mathbb{R})$,

$$\begin{aligned}
[(W_N \otimes W_M)\mathcal{S}(f)](x + yM) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (W_N \otimes W_M)(x + yM, m + nM) \mathcal{S}(f)(m + nM) \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_N(y, n) W_M(x, m) f(m, n) \\
&= \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{2\pi j (\frac{mx}{M} + \frac{ny}{N})} \\
&= \sqrt{MN} \left[ \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \overline{e^{-2\pi j (\frac{mx}{M} + \frac{ny}{N})}} \right] \\
&= \sqrt{MN}\, \overline{DFT(f)(x, y)}
\end{aligned}$$

$$= \sqrt{MN}\,\overline{\mathcal{S}(DFT(f))(x+yM)}.$$

Hence $(W_N \otimes W_M)\mathcal{S}(f) = \sqrt{MN}\mathcal{S}(\overline{DFT(f)})$.

(e) Recall the result from (a):

$$(\lambda D^T D + L^T L)\vec{f} = \lambda D^T \vec{g}.$$

Since $D$ and $L$ are block-circulant, the result from (c)ii. asserts that

$$[(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})](x+kM, y+lM) = \begin{cases} MN\overline{DFT(h)(x,k)} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise} \end{cases};$$

equivalently, denoting the $MN \times MN$ diagonal matrix by $\Lambda_D$,

$$D = (\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M).$$

Similarly,

$$L = (\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M).$$

Hence

$$\begin{aligned}
(\lambda D^T D + L^T L)\vec{f} &= (\lambda D^* D + L^* L)\vec{f} \\
&= \{\lambda[(\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M)]^*(\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M) \\
&\quad + [(\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M)]^*(\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M)\}\vec{f} \\
&= [\lambda(W_N \otimes W_M)^*\Lambda_D^*\Lambda_D(W_N \otimes W_M) + (W_N \otimes W_M)^*\Lambda_L^*\Lambda_L(W_N \otimes W_M)]\vec{f} \\
&= (\overline{W_N} \otimes \overline{W_M})(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)(W_N \otimes W_M)\vec{f} \\
&= (\overline{W_N} \otimes \overline{W_M})(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\sqrt{MN}\mathcal{S}(\overline{DFT(f)});
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\lambda D^T \vec{g} &= \lambda D^* \vec{g} \\
&= \lambda(W_N \otimes W_M)^*\Lambda_D^*(\overline{W}_N \otimes \overline{W_M})^*\vec{g} \\
&= \lambda(\overline{W_N} \otimes \overline{W_M})\Lambda_D^*(W_N \otimes W_M)\vec{g} \\
&= \lambda(\overline{W_N} \otimes \overline{W_M})\Lambda_D^*\sqrt{MN}\,\mathcal{S}(\overline{DFT(g)}).
\end{aligned}$$

Hence $(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\mathcal{S}(\overline{DFT(f)}) = \lambda\Lambda_D^*\mathcal{S}(\overline{DFT(g)})$. By comparing each pair of entries,

$$(\lambda M^2N^2|DFT(h)(u,v)|^2 + M^2N^2|DFT(p)(u,v)|^2)\overline{DFT(f)(u,v)} = \lambda MNDFT(h)(u,v)\overline{DFT(g)(u,v)},$$

which yields

$$DFT(f)(u,v) = \frac{\lambda\overline{DFT(h)(u,v)}DFT(g)(u,v)}{MN(\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2)}.$$

7. To be supplemented.

8. To be supplemented.