# mmat5390: mathematical image processing
## assignment 1 solutions

1. (a) Note that $H$ is a $4 \times 4$ matrix; hence it represents a linear transformation on $2 \times 2$ images.
   $H$ is not block-circulant. For example, consider the $y = 2, \beta = 2$-submatrix of $H$, i.e. $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, which is not a circulant matrix, as the shift-operator $t$ maps $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} b \\ a \end{pmatrix}$ instead of $\begin{pmatrix} c \\ d \end{pmatrix}$, which is different from $\begin{pmatrix} b \\ a \end{pmatrix}$ since $b \neq c$.
   Hence $h$ is not shift-invariant with $h_s$ being 2-periodic in both arguments.
   Furthermore, $H$ is not block-toeplitz, thus $h$ is neither shift-invariant.

   $H$ is not a kronecker product of two $2 \times 2$ matrices. for example, consider the $y = 1, \beta = 1$- and $y = 2, \beta = 2$-submatrices of $h$, i.e. $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ and $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. neither is a scalar multiple of the other. hence $h$ is not separable.

   (b) Note that $H$ is a $9 \times 9$ matrix; hence it represents a linear transformation on $3 \times 3$ images.
   $H$ is not block-circulant. For example, consider the $y = 1, \beta = 2$-submatrix of $h$, i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}$, which is not a circulant matrix, as the shift-operator $t$ maps $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ to $\begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}$ instead of $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$. Hence $h$ is not shift-invariant with $h_s$ being 3-periodic in both arguments. (Neither is $H$ block-toeplitz, hence neither is $h$ shift-invariant.)
   $H$ is the kronecker product of two $3 \times 3$ matrices; explicitly,

$$H = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix}.$$

   Hence $h$ is separable.

   (c) Note that $H$ is a $4 \times 4$ matrix; hence it represents a linear transformation on $2 \times 2$ images.
   $H$ is block-circulant. The $y = 1, \beta = 1$- and the $y = 2, \beta = 2$-submatrices of $H$ are both $\begin{pmatrix} \pi & 2\pi \\ 2\pi & \pi \end{pmatrix}$, which is circulant; the $y = 2, \beta = 1$- and the $y = 1, \beta = 2$-submatrices of $h$ are both $\begin{pmatrix} 3\pi & 4\pi \\ 4\pi & 3\pi \end{pmatrix}$, which is also circulant. hence $h$ is shift-invariant with $h_s$ being 2-periodic in both arguments.
   $H$ is not a kronecker product of two $2 \times 2$ matrices. for example, consider the $y = 1, \beta = 1$- and $y = 2, \beta = 1$-submatrices of $H$, i.e. $\begin{pmatrix} \pi & 2\pi \\ 2\pi & \pi \end{pmatrix}$ and $\begin{pmatrix} 3\pi & 4\pi \\ 4\pi & 3\pi \end{pmatrix}$. neither is a scalar multiple of the other. Hence $h$ is not separable.

   (d) Note that $H$ is a $9 \times 9$ matrix; hence it represents a linear transformation on $3 \times 3$ images.
   Obviously, $H$ is block-circulant. Denote $H = (A_{ij})$, then
   $A_{11} = A_{22} = A_{33} = \begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$, $A_{12} = A_{23} = A_{31} = \begin{pmatrix} 18 & 18 & 36 \\ 36 & 18 & 18 \\ 18 & 36 & 18 \end{pmatrix}$

and $A_{13} = A_{21} = A_{32} = \begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$. All these three matrices are circulant.

$H$ is the kronecker product of two $3 \times 3$ matrices; explicitly,

$$H = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}.$$

Hence $h$ is separable.

2. let $h$ be the separable psf of a linear image transformation, with $h(x, \alpha, y, \beta) = h_c(x, \alpha)h_r(y, \beta)$. let $h$ be the corresponding transformation matrix.

   then the $y = k, \beta = l$-submatrix of $h$ (denoted by $\tilde{h}_{kl}$) is given by

$$\begin{aligned}
\left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = k \\ \beta = l \end{pmatrix}} \right) &= [h\,(\alpha + (l-1)n, x + (k-1)n)]_{\substack{1 \le x \le n \\ 1 \le \alpha \le n}} \\
&= [h(x, \alpha, k, l)]_{\substack{1 \le x \le n \\ 1 \le \alpha \le n}} \\
&= [h_c(x, \alpha)h_r(k, l)]_{\substack{1 \le x \le n \\ 1 \le \alpha \le n}} \\
&= h_r(k, l)[h_c(x, \alpha)]_{\substack{1 \le x \le n \\ 1 \le \alpha \le n}} \\
&= h_r(k, l)h_c^T.
\end{aligned}$$

   recall that

$$h = \begin{pmatrix}
\left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 1 \\ \beta = 1 \end{pmatrix}} \right) & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 2 \\ \beta = 1 \end{pmatrix}} \right) & \cdots & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = n \\ \beta = 1 \end{pmatrix}} \right) \\
\left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 1 \\ \beta = 2 \end{pmatrix}} \right) & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 2 \\ \beta = 2 \end{pmatrix}} \right) & \cdots & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = n \\ \beta = 2 \end{pmatrix}} \right) \\
\vdots & \vdots & \ddots & \vdots \\
\left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 1 \\ \beta = n \end{pmatrix}} \right) & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = 2 \\ \beta = n \end{pmatrix}} \right) & \cdots & \left( \alpha \downarrow \overset{x \rightarrow}{\begin{pmatrix} y = n \\ \beta = n \end{pmatrix}} \right)
\end{pmatrix}$$

$$= \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{21} & \cdots & \tilde{h}_{n1} \\ \tilde{h}_{12} & \tilde{h}_{22} & \cdots & \tilde{h}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{1n} & \tilde{h}_{2n} & \cdots & \tilde{h}_{nn} \end{pmatrix} = \begin{pmatrix} h_r(1,1)h_c^T & h_r(2,1)h_c^T & \cdots & h_r(n,1)h_c^T \\ h_r(1,2)h_c^T & h_r(2,2)h_c^T & \cdots & h_r(n,2)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r(1,n)h_c^T & h_r(2,n)h_c^T & \cdots & h_r(n,n)h_c^T \end{pmatrix}$$

$$= \begin{pmatrix} h_r^T(1,1)h_c^T & h_r^T(1,2)h_c^T & \cdots & h_r^T(1,n)h_c^T \\ h_r^T(2,1)h_c^T & h_r^T(2,2)h_c^T & \cdots & h_r^T(2,n)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r^T(n,1)h_c^T & h_r^T(n,2)h_c^T & \cdots & h_r^T(n,n)h_c^T \end{pmatrix} = h_r^T \otimes h_c^T.$$

3. let $f, g \in M_{m \times n}(\mathbb{R})$, and assume that they are periodically extended.

let $\alpha \in \mathbb{N} \cap [1, m]$ and $\beta \in \mathbb{N} \cap [1, n]$. by definition,

$$
\begin{aligned}
f * g(\alpha, \beta) &= \sum_{x=1}^{m} \sum_{y=1}^{n} f(x, y) g(\alpha - x, \beta - y) \\
&= \sum_{i=\alpha-m}^{\alpha-1} \sum_{j=\beta-n}^{\beta-1} f(\alpha - i, \beta - j) g(i, j) \text{ (letting } i = \alpha - x,\ j = \beta - y) \\
&= \sum_{i=\alpha-m}^{0} \sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j) g(i, j) + \sum_{i=\alpha-m}^{0} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i, j) \\
&+ \sum_{i=1}^{\alpha-1} \sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j) g(i, j) + \sum_{i=1}^{\alpha-1} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i, j) \\
&= \sum_{i=\alpha}^{m} \sum_{j=\beta}^{n} f(\alpha - i, \beta - j) g(i, j) + \sum_{i=\alpha}^{m} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i, j) \\
&+ \sum_{i=1}^{\alpha-1} \sum_{j=\beta}^{n} f(\alpha - i, \beta - j) g(i, j) + \sum_{i=1}^{\alpha-1} \sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j) g(i, j) \text{ (by periodicity)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} g(i, j) f(\alpha - i, \beta - j) \\
&= g * f(\alpha, \beta);
\end{aligned}
$$

hence $f * g = g * f$.

4. Since $\mathcal{O}$ is a shift-invariant linear image transformation on $M_{n\times n}(\mathbb{R})$ with $h_s(\cdot, \cdot)$ being n-periodic in both arguments, by Theorem 3.3 in Chapter 1, we have

$$
H = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}
$$

where

$$
\begin{aligned}
A_{ij} &= \begin{pmatrix} h_s(0, i-j) & h_s(-1, i-j) & \cdots & h_s(1-n, i-j) \\ h_s(1, i-j) & h_s(0, i-j) & \cdots & h_s(2-n, i-j) \\ \vdots & \vdots & \ddots & \vdots \\ h_s(n-1, i-j) & h_s(n-2, i-j) & \cdots & h_s(0, i-j) \end{pmatrix} \\
&= \begin{pmatrix} h_s(0, i-j) & h_s(n-1, i-j) & \cdots & h_s(1, i-j) \\ h_s(1, i-j) & h_s(0, i-j) & \cdots & h_s(2, i-j) \\ \vdots & \vdots & \ddots & \vdots \\ h_s(n-1, i-j) & h_s(n-2, i-j) & \cdots & h_s(0, i-j) \end{pmatrix}
\end{aligned}
$$

Suppose $A_{ij}$ is n-periodic extended, i.e. $A_{(i+n)(j+n)} = A_{ij}$. Since $h_s(\cdot, \cdot)$ is n-periodic in the second argument, we know that for any $k \in \mathbb{N}$,

$$
\begin{aligned}
A_{(i+k)(j+k)} &= \begin{pmatrix} h_s(0, (i+k)-(j+k)) & h_s(n-1, (i+k)-(j+k)) & \cdots & h_s(1, (i+k)-(j+k)) \\ h_s(1, (i+k)-(j+k)) & h_s(0, (i+k)-(j+k)) & \cdots & h_s(2, (i+k)-(j+k)) \\ \vdots & \vdots & \ddots & \vdots \\ h_s(n-1, (i+k)-(j+k)) & h_s(n-2, (i+k)-(j+k)) & \cdots & h_s(0, (i+k)-(j+k)) \end{pmatrix} \\
&= \begin{pmatrix} h_s(0, i-j) & h_s(n-1, i-j) & \cdots & h_s(1, i-j) \\ h_s(1, i-j) & h_s(0, i-j) & \cdots & h_s(2, i-j) \\ \vdots & \vdots & \ddots & \vdots \\ h_s(n-1, i-j) & h_s(n-2, i-j) & \cdots & h_s(0, i-j) \end{pmatrix} = A_{ij}
\end{aligned}
$$

Therefore, $H = \begin{pmatrix} A_{11} & A_{n1} & \cdots & A_{21} \\ A_{21} & A_{11} & \cdots & A_{31} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{(n-1)1} & \cdots & A_{11} \end{pmatrix}$ with $A_{ij}$ being circulant matrix. $H$ is block-circilant.

5. (a)

$$
\begin{aligned}
|A - \alpha B|_F^2 &= \left| \begin{pmatrix} 3-\alpha & 3-\alpha & 6-\alpha \\ -\alpha & 1-\alpha & 2-\alpha \\ -\alpha & 1-\alpha & 8-\alpha \end{pmatrix} \right|_F^2 \\
&= 2\alpha^2 + 2(1-\alpha)^2 + (2-\alpha)^2 + 2(3-\alpha)^2 + (6-\alpha)^2 + (8-\alpha)^2 \\
&= 9\alpha^2 - 48\alpha + 124 = 9(\alpha - \frac{24}{9})^2 + 60.
\end{aligned}
$$

hence $|A - \alpha B|_F$ is minimized at $\alpha = \dfrac{24}{9}$.

(b)

$$
\begin{aligned}
|C - \alpha D|_F^2 &= \left| \begin{pmatrix} 2-\alpha & 3-\alpha & 5-\alpha & 7-\alpha \\ 8-\alpha & 6-\alpha & 4-\alpha & 2-\alpha \end{pmatrix} \right|_F^2 \\
&= 2(2-\alpha)^2 + (3-\alpha)^2 + (4-\alpha)^2 + (5-\alpha)^2 + (6-\alpha)^2 + (7-\alpha)^2 + (8-\alpha)^2 \\
&= 8\alpha^2 - 74\alpha + 207 = 8(\alpha - \frac{37}{8})^2 + \frac{287}{8}.
\end{aligned}
$$

hence $|C - \alpha D|_F$ is minimized at $\alpha = \dfrac{37}{8}$.

(c) The values of $\alpha$ that minimize the frobenius norm differences are the means of the pixel values. The values of $\alpha$ that minimize the entrywise 1-norm differences are the medians of the pixel values.

6. (a) we first compute the characteristic polynomial of $A^TA$ to obtain the singular values of $A$.

$$
A^TA = \begin{pmatrix} 10 & 8 & 6 \\ 8 & 8 & 8 \\ 6 & 8 & 10 \end{pmatrix}.
$$

Hence the characteristic polynomial of $A^TA$ is given by

$$
\begin{aligned}
\det(A^TA - \lambda I_3) &= \begin{vmatrix} 10-\lambda & 8 & 6 \\ 8 & 8-\lambda & 8 \\ 6 & 8 & 10-\lambda \end{vmatrix} \\
&= -\lambda^3 + 28\lambda^2 - 96\lambda \\
&= -\lambda(\lambda-4)(\lambda-24).
\end{aligned}
$$

The singular values of $A$ are given to be integers. Then one solves for the eigenvector corresponding to each eigenvalue of $A^TA$.

For $\lambda_1 = 24$,

$$
[A^TA - \lambda_1 I_3 | 0] = \left[ \begin{array}{ccc|c} -14 & 8 & 6 & 0 \\ 8 & -16 & 8 & 0 \\ 6 & 8 & -14 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],
$$

So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_1 = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 4$,

$$[A^TA - \lambda_2 I_3|0] = \left[\begin{array}{ccc|c} 6 & 8 & 6 & 0 \\ 8 & 4 & 8 & 0 \\ 6 & 8 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_2 = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda_3 = 0$,

$$[A^TA - \lambda_3 I_3|0] = \left[\begin{array}{ccc|c} 10 & 8 & 6 & 0 \\ 8 & 8 & 8 & 0 \\ 6 & 8 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_3 = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Then $\vec{u}_1 = \dfrac{1}{\sqrt{\lambda_1}}A\vec{v}_1 = \dfrac{1}{\sqrt{24}}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$\vec{u}_2 = \dfrac{1}{\sqrt{\lambda_2}}A\vec{v}_2 = \dfrac{1}{\sqrt{4}}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

Hence an svd of $A$ is given by $A = U\Sigma V^T$, where $U = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sqrt{24} & 0 & 0 \\ 0 & \sqrt{4} & 0 \end{pmatrix}$ and $V = \dfrac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}$.

(b) The elementary images according to the above svd are given by:

$$\begin{aligned} \vec{u}_1\vec{v}_1^T &= \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}; \\ \vec{u}_2\vec{v}_2^T &= \frac{1}{\sqrt{4}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & 0 & -\frac{1}{\sqrt{4}} \end{pmatrix}; \end{aligned}$$

Hence,

$$\begin{aligned} A &= \sqrt{24}\vec{u}_1\vec{v}_1^T + \sqrt{8}\vec{u}_2\vec{v}_2^T \\ &= \sqrt{24}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} + \sqrt{8}\begin{pmatrix} -\frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & 0 & -\frac{1}{\sqrt{4}} \end{pmatrix} \end{aligned}$$

7. (a) As $A = U\Sigma V^T$,

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$$

and

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T.$$

Note that

$$\Sigma^T\Sigma = \begin{cases} \begin{pmatrix} \begin{matrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{matrix} & \mathbf{0}_{M\times(N-M)} \\ \mathbf{0}_{(N-M)\times M} & \mathbf{0}_{(N-M)\times(N-M)} \end{pmatrix} & \text{if } M < N \\ \begin{pmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{pmatrix} & \text{if } M \geq N \end{cases}$$

and

$$\Sigma\Sigma^T = \begin{cases} \begin{pmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{pmatrix} & \text{if } M \leq N \\ \begin{pmatrix} \begin{matrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{matrix} & \mathbf{0}_{N\times(M-N)} \\ \mathbf{0}_{(M-N)\times N} & \mathbf{0}_{(M-N)\times(M-N)} \end{pmatrix} & \text{if } M > N \end{cases}$$

Hence $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$ are the square roots of the largest $K$ eigenvalues of $A^TA$ (or $AA^T$) in descending order, and thus the $K$-tuple is uniquely determined.

(b) Suppose $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ are distinct and nonzero. Then each eigenspace of $A^TA$ and $AA^T$ corresponding to eigenvalue $\sigma_{ii}^2$ has dimension 1, which means that there are exactly two unit eigenvectors to be chosen from each eigenspace, each being the negative of the other. Such eigenvectors are precisely the first $K$ columns of $U$ and $V$. Combined with the fact that $\sigma_{ii}$ are in descending order, the first $K$ columns of $U$ and $V$ are uniquely determined up to a change of sign.

(c) A counterexample with nondistinct $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ is given by:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 I_2 I_2 = U I_2 U^T,$$

where $I_2$ and $U = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ are unitary.

A counterexample with $\sigma_{KK} = 0$ is given by:

$$(0\ \ 0) = (1)(0\ \ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(0\ \ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$