Lecture 9:

Speeding up Beltrami Holomorphic Flow

Key idea of BHF: Given $f^{\mu}: \Omega_1 \to \Omega_2$ with Beltrami coefficient μ, find f^{ν} with B.C. ν.

Let $\{\mu_n\}_{n=0}^{\infty} \ni \mu_0 = \mu$ and $\mu_\infty = \nu$. Proceed to modify f^{μ} iteratively:

$$\begin{array}{ccccccccc}
\mu_0 = \mu & \to & \mu_1 & \to & \cdots & \to & \mu_n & \to \cdots \to & \mu_\infty = \nu \\
\updownarrow & & & & & & & & \\
f^{\mu_0} := f^{\mu} & \to & f^{\mu_1} & \to & \cdots & \to & f^{\mu_n} & \to \cdots \to & f^{\mu_\infty} = f^{\nu}
\end{array}$$

For a small variation from μ_0 to μ_1, we solve for $f^{\mu_1} \ni$

$$f^{\mu_1} = \operatorname{argmin}_{f \in \underbrace{\mathrm{Diff}}_{\text{Space of diffeomorphism}}} \left\{ \left\| \frac{\partial f}{\partial \bar{z}} - \mu_1 \frac{\partial f}{\partial z} \right\|_2^2 \right\}.$$

Subject to $f^{\mu_1}\big|_{\partial\Omega_1} = \partial\Omega_2$

Let $f^{\mu_1} = f^{\mu_0} + g_1$. Reformulate as finding $g_1 : \Omega_1 \to \mathbb{R}^2 \ni$

$$(*) \quad g_1 = \operatorname*{argmin}_{g:\Omega_1 \to \mathbb{R}^2} \left\{ \left\| \frac{\partial (f^{\mu_0} + g)}{\partial \bar{z}} - \mu_1 \frac{\partial (f^{\mu_0} + g)}{\partial z} \right\|_2^2 \right\}$$

$$= \operatorname*{argmin}_{g:\Omega_1 \to \mathbb{R}^2} \left\{ \left\| A(\mu_1) g + A(\mu_2) f^{\mu_0} \right\|_2^2 \right\} \quad (*)$$

subject to boundary constraint.

where $A(\mu_1) \overset{def}{:=} \frac{\partial}{\partial \bar{z}} - \mu_1 \frac{\partial}{\partial z}$.

Discrete

Least Square Problem

Fast!!

(Instead of computing integration)

Algorithm 1 : *(Beltrami holomorphic flow)*
Input : $f^{\mu} : \Omega_1 \to \Omega_2$ *with BC* μ, *target BC* ν, *threshold* ϵ'
Output : *Sequence of quasi-conformal maps* $\{f^{\mu_n}\}_{n=1}^{\infty}$

1. *Set* $f^{\mu_0} = f^{\mu}$. *Solve Equation* $(*)$ *to obtain* g_1;
2. *Given* f^{μ_n}, *compute* $\mu_n := \mu(f_n)$ *and* $\nu_n := (1-\epsilon)\mu_n + \epsilon\nu$; *solve Equation* $(*)$ *to obtain* g_{n+1}; *Set* $f_{n+1} := f_n + g_{n+1}$;
3. *If* $||\mu_{n+1} - \mu_n|| \geq \epsilon'$, *repeat step 2. Otherwise, stop the iteration.*

ADMM + BHF to solve Diffeomorphism Optimization Problem

Background: Minimize $\{ \underbrace{E_1(\overset{\in \mathbb{R}^n}{x}) + E_2(Ax)}_{\text{usually convex.}} \}$ where $A \in M_{m\times n}(\mathbb{R})$ has full column rank.

Reformulate: $(*)$ Minimize $\{ E_1(x) + E_2(y) \}$ Subject to $Ax = y$.

Then: the augmented Lagrangian is given by:

$$L(x, y, \lambda, \mu) = E_1(x) + E_2(y) + \lambda^T (Ax - y) + \frac{\mu_k}{2} \| Ax - y \|^2$$

$(*)$ can be solved by:

$$\begin{cases} (x^{k+1}, y^{k+1}) = \operatorname{argmin} \{ L(x, y, \lambda^k, \mu^k) \} \\ \lambda^{k+1} = \lambda^k + \mu_k (A x^{k+1} - y^{k+1}) \end{cases}$$

$\star$

Different choices of $\{\mu_k\}$ has been proposed to ensure convergence.

$\uparrow$ Penalty term

(*) can be difficult to solve.

Alternating direction method with multiplier (ADMM):

$$\begin{cases} x^{k+1} = \operatorname*{argmin}_x \{ L(x, y^k, \lambda^k, \mu_k) \} \\ y^{k+1} = \operatorname*{argmin}_y \{ L(x^{k+1}, y, \lambda^k, \mu_k) \} \\ \lambda^{k+1} = \lambda^k + \mu_k (Ax^{k+1} - y^{k+1}) \end{cases}$$

Remark: Once $\{\lambda^k\}$ and $\{\mu_k\}$ are carefully chosen, ADMM minimizes in few iterations.

Now, suppose we need to solve Diffeomorphism Optimization Problem:

$$f^* = \underset{f \in \mathrm{Diff}}{\operatorname{argmin}} \{ E_1(f) + E_2(\mu(f)) \} \text{ subject to:}$$

$$\|\mu(f^*)\|_\infty \overset{\text{def}}{=} \left\| \left(\frac{\partial f^*}{\partial \bar{z}}\right) \Big/ \left(\frac{\partial f^*}{\partial z}\right) \right\|_\infty < 1 .$$

e.g. Find $f^*: S_1 \to S_2$ such that it minimizes:

$$E(f) = \int_{S_1} |\mu(f)|^p + \alpha \int_{S_1} |H_1 - H_2(f)|^2$$

mean curvatures

Reformulate: Find $f^*: S_1 \to S_2$ and $\nu^*: S_1 \to \mathbb{C}$ $\ni$

$$(f^*, \nu^*) = \underset{\substack{f \in \mathrm{Diff} \\ \mu \in \mathcal{B}}}{\operatorname{argmin}} \{ E_1(f) + E_2(\mu) \} \ni \text{ (1) } \nu^* = \mu(f^*)$$

$$\text{(2) } \|\nu^*\|_\infty < 1 .$$

Space of Beltrami coefficient

Augmented Lagrangian:

$$L(f, \nu, \lambda_{Re}, \lambda_{Im}, \rho) = E_1(f) + E_2(\nu) + \langle \lambda_{Re}, Re(\nu - \mu(f)) \rangle + \langle \lambda_{Im}, Im(\nu - \mu(f)) \rangle + \frac{\rho}{2} \|\nu - \mu(f)\|_2^2$$

where $\langle \alpha, \beta \rangle \overset{def}{=} \int_{S_1} \alpha \beta$ and $\|\alpha\| \overset{def}{=} \left(\int_{S_1} |\alpha|^2 \right)^{1/2}$

Using ADMM:

$$f^{k+1} = \underset{f}{\operatorname{argmin}} \{ L(f, \nu_k, \lambda_{Re}^k, \lambda_{Im}^k, \rho^k) \} \quad — \quad 1$$

$$\nu^{k+1} = \underset{\nu}{\operatorname{argmin}} \{ L(f_{k+1}, \nu, \lambda_{Re}^k, \lambda_{Im}^k, \rho^k) \} \quad — \quad 2$$

λ_{Re}^k, λ_{Im}^k and ρ^k are updated as follows:

if $\|\nu^{k+1} - \mu(f^{k+1})\|_2 < \eta_k$, update:

$$\lambda_{Re}^{k+1} = \lambda_{Re}^k + \rho_k Re(\nu^{k+1} - \mu(f^{k+1}))$$

$$\lambda_{Im}^{k+1} = \lambda_{Im}^k + \rho_k Im(\nu^{k+1} - \mu(f^{k+1}))$$

$$\rho_{k+1} = \rho_k$$

if $\| \nu^{k+1} - \mu(f^{k+1}) \| \geq \eta_k$, update:

$$\lambda_{Re}^{k+1} = \lambda_{Re}^{k} \; ; \; \lambda_{Im}^{k+1} = \lambda_{Im}^{k}$$

$$\rho_{k+1} = \rho_k (1 + \gamma_k)$$

(η_k is chosen to be $\downarrow$
γ_k can be constant)

<u>Solving subproblem (2) involving ν:</u>

Very often, Euler-Lagrange eqt of $E_2(\mu)$ is an elliptic PDE. e.g. $\int_{S_1} |\nabla \mu|^2 + |\mu|^2$. Then, E-L eqt of (2) can be written as:

$$\underbrace{\Delta \mu - 2\mu - \lambda_{Re}^{k} Re(\nu - \mu(f^{k+1})) - i \lambda_{Im}^{k} Im(\nu - \mu(f^{k+1})) - \rho_k (\nu - \mu(f^{k+1})) = 0}$$

Discretize $\rightsquigarrow$ Sparse Symmetric positive definite linear system.

Solving subproblem ① involving f:

$$f^{k+1} = \underset{f}{\operatorname{argmin}} \left\{ E_1(f) + \langle \lambda_{Re}, Re(\nu - \mu(f)) \rangle + \langle \lambda_{Im}, Im(\nu - \mu(f)) \rangle + \frac{\rho}{2} \|\nu - \mu(f)\|_2^2 \right\}$$

$E_1(f)$ can be minimized using gradient descent algorithm.

e.g. $E_1(f) = \| I_1 - I_2(f) \|_2^2$

curvatures on S_1 and S_2

Then: descent direction $\vec{V}_1 = 2(I_1 - I_2(f)) \nabla f$

Last three terms can be minimized over B.C. to get a descent direction $\partial \tilde{\nu}$:

$$\partial \tilde{\nu} = -\lambda_{Re} - i \lambda_{Im} + \rho(\nu - \mu(f))$$

After few iteration, we get a new B.C. $\tilde{\nu}$.

We can find the associated QC map $\tilde{f} = f^k + \vec{V}_2 \Rightarrow$

$$\frac{\partial \tilde{f}}{\partial \bar{z}} \approx \tilde{\nu} \frac{\partial \tilde{f}}{\partial z}$$

Overall, the descent direction to optimize:

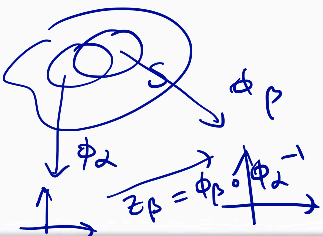
$$\frac{df^k}{dt} = \vec{V}_1 + \vec{V}_2$$

Computation of QC map using auxiliary metric

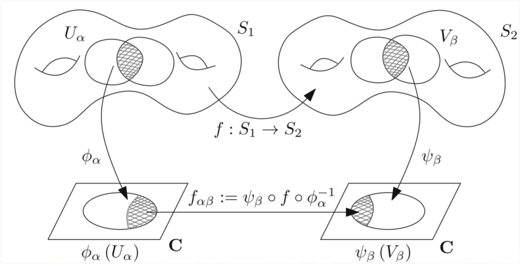
Definition: (Beltrami Differential) A Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ on a Riemann surface S is an assignment to each chart (U_α, ϕ_α) of an L^∞ complex-valued function μ_α defined on local parameters z_α such that:

$$\mu_\alpha(z_\alpha)\frac{dz_\beta}{dz_\alpha} = \mu_\beta(z_\beta)\frac{d\overline{z_\beta}}{d\overline{z_\alpha}}$$

on the domain which is also covered by another chart (U_β, z_β), where $\frac{dz_\beta}{dz_\alpha} = \frac{d}{dz_\alpha}\phi_{\alpha\beta}$ and $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$.



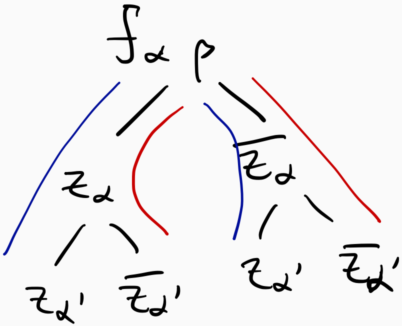
Definition: (QC map between Riemann surfaces) An orientation-preserving homeomorphism $f: S_1 \to S_2$ is called quasi-conformal associated with $\mu \frac{d\bar{z}}{dz}$ if for any chart (U_α, ϕ_α) on S_1 and for any chart (V_β, ψ_β) on S_2, the mapping $f_{\alpha\beta} := \psi_\beta \circ f \circ \phi_\alpha^{-1}$ is QC associated with $\mu_\alpha(z_\alpha)$. Also, on the domain on S_1 which is also covered by $(U_{\alpha'}, \phi_{\alpha'})$, $f_{\alpha'\beta} := \psi_\beta \circ f \circ \phi_{\alpha'}^{-1}$ is QC associated with $\mu_{\alpha'}(z_{\alpha'})$ where $\mu_{\alpha'}(z_{\alpha'}) = \mu_\alpha(z_\alpha)\left(\frac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right) \Big/ \left(\frac{dz_\alpha}{dz_{\alpha'}}\right)$.



Well-defined?

- On region covered by two different charts z_α and $z_{\alpha'}$, we have $\dfrac{dz_\alpha}{d\bar{z}_{\alpha'}} \overset{\text{def}}{=} \dfrac{d}{d\bar{z}_{\alpha'}} \underbrace{\phi_\alpha \circ \phi_{\alpha'}^{-1}}_{\text{holomorphic}} = 0.$

$$\therefore\ \mu_{\alpha'}(z_{\alpha'}) = \left(\frac{\partial f_{\alpha'\beta}}{\partial \bar{z}_{\alpha'}}\right) \Big/ \left(\frac{\partial f_{\alpha'\beta}}{\partial z_{\alpha'}}\right) = \frac{\left(\dfrac{\partial f_{\alpha\beta}}{\partial z_\alpha} \overset{0}{\cancel{\dfrac{dz_\alpha}{d\bar{z}_{\alpha'}}}} + \dfrac{\partial f_{\alpha\beta}}{\partial \bar{z}_\alpha} \dfrac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right)}{\left(\dfrac{\partial f_{\alpha\beta}}{\partial z_\alpha} \dfrac{dz_\alpha}{dz_{\alpha'}} + \dfrac{\partial f_{\alpha\beta}}{\partial \bar{z}_\alpha} \overset{0}{\cancel{\dfrac{d\bar{z}_\alpha}{dz_{\alpha'}}}}\right)}$$



$$= \frac{\left(\dfrac{\partial f_{\alpha\beta}}{\partial \bar{z}_\alpha}\right)}{\left(\dfrac{\partial f_{\alpha\beta}}{\partial z_\alpha}\right)} \frac{\overline{\left(\dfrac{dz_\alpha}{dz_{\alpha'}}\right)}}{\left(\dfrac{dz_\alpha}{dz_{\alpha'}}\right)}, \qquad \overline{\left(\dfrac{dz_\alpha}{dz_{\alpha'}}\right)} = \left(\frac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right)$$

$$= \mu_\alpha(z_\alpha) \frac{\left(\dfrac{d\bar{z}_\alpha}{d\bar{z}_{\alpha'}}\right)}{\left(\dfrac{dz_\alpha}{dz_{\alpha'}}\right)}$$

Check: $\dfrac{\partial}{\partial z}\phi = \overline{\left(\dfrac{\partial}{\partial \bar{z}}\bar{\phi}\right)}$

- Let ψ_β and $\psi_{\beta'}$ be two different charts on the range of f, μ_β and $\mu_{\beta'}$ be the Beltrami coefficient computed under $f_{\alpha\beta}$ and $f_{\alpha\beta'}$ resp. Then:

$$f_{\alpha\beta'} = \psi_{\beta'} \circ f \circ \phi_\alpha^{-1} = \underbrace{\psi_{\beta'} \circ \psi_\beta^{-1}}_{\text{conf}} \circ \underbrace{\psi_\beta \circ f \circ \phi_\alpha^{-1}}_{f_{\alpha\beta}}$$

By composition formula:

$$\mu_{g\circ f} = \frac{\mu_f + (\mu_g \circ f)\tau}{1 + \overline{\mu_f}(\mu_g \circ f)\tau}.$$

If g is conf, then $\mu_g \equiv 0$.

$$\therefore \boxed{\mu_\beta = \mu_{\beta'}}$$

Theorem: (Auxilliary metric associated with a Beltrami Differential) Suppose (S_1, g_1) and (S_2, g_2) are two metric surfaces, $f: S_1 \to S_2$ is a QC map associated with the Beltrami differential $\mu \frac{d\bar{z}}{dz}$. Let z and w be the local isothermal coordinates of S_1 and S_2 respectively, indeed $g_1 = e^{2\lambda_1(z)} dz\, d\bar{z}$ and $g_2 = e^{2\lambda_2(w)} dw\, d\bar{w}$. Define an auxiliary Riemannian metric on S_1,

$$\tilde{g}_1 = e^{2\lambda_1(z)} |dz + \mu\, d\bar{z}|^2.$$

Then: the mapping $f: (S_1, \tilde{g}_1) \to (S_2, g_2)$ is a conformal mapping.