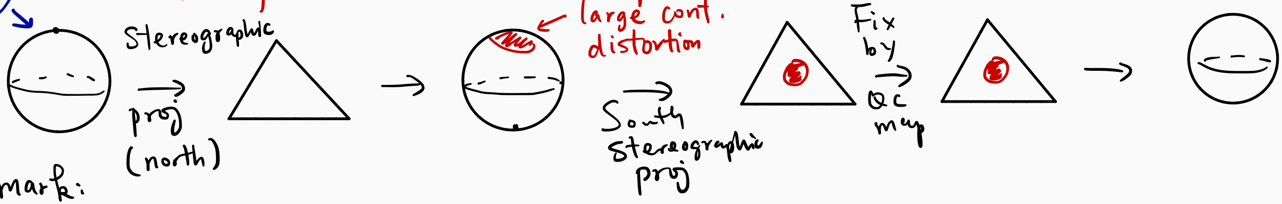
Lecture 8:

Recap: Last time, fix conformality distortion for "Fast algorithm of spherical conformal parameterization" using QC map



Remark:

1. Landmark-matching optimized spherical harmonic parameterization can be improved similarly.
2. Fixing non-bijectivity if $f: M \to N$ is non-bijective, then: $|\mu(f)(z)| \geq 1$ at some z, then we can fix it by truncation:

$$\tilde{\mu}(f)(z) = \begin{cases} \tilde{\mu}(z) & \text{if } |\tilde{\mu}(z)| \leq 1-\varepsilon \\ (1-\varepsilon)\arg(\tilde{\mu}(z)) & \text{if } |\tilde{\mu}(z)| \geq 1 \end{cases}$$

Beltrami Holomorphic Flow (Bojarski)

Theorem: (Beltrami Holomorphic flow on $\mathbb{S}^2$) There is a one-to-one correspondence between the set of quasiconformal diffeomorphisms of $\mathbb{S}^2$ that fix the points 0, 1 and ∞ and the set of smooth complex-valued functions μ on $\mathbb{S}^2$ with $\|\mu\|_\infty = k < 1$.

(Here, we identify $\mathbb{S}^2$ with extended complex plane $\overline{\mathbb{C}}$).

Also, the solution to the Beltrami's eqt depends holomorphically on μ. Let $\{\mu(t)\}$ be a family of Beltrami coefficient depending on a real / complex t. Let $\mu(t)(z) = \mu(z) + t\nu(z) + t\varepsilon(t)(z)$ for $z \in \mathbb{C}$ and $\|\varepsilon(t)\|_\infty \to 0$ as $t \to 0$. Then: for $\forall w \in \mathbb{C}$, $\nu, \varepsilon(t) \in L^\infty(\mathbb{C})$

$$f^{\mu(t)}(w) = f^{\mu}(w) + t\,V(f^\mu, \nu)(w) + o(|t|) \text{ locally uniformly on } \mathbb{C}$$

as $t \to 0$, where

$$V(f^\mu, \nu)(w) = \frac{-f^\mu(w)(f^\mu(w)-1)}{\pi} \int_{\mathbb{C}} \frac{\nu(z)\left((f^\mu)_z(z)\right)^2}{f^\mu(z)(f^\mu(z)-1)(f^\mu(z)-f^\mu(w))}\,dx\,dy$$

BHF algorithm Let M = genus-0 mesh

N = genus-0 mesh.

Let $\phi_1 : M \to \mathbb{S}^2 \cong \overline{\mathbb{C}}$ and $\phi_2 : N \to \mathbb{S}^2 \cong \overline{\mathbb{C}}$ be global conformal parameterization of M and N respectively.

Given $\mu : M \to \mathbb{C}$, consider $\tilde{\mu} = \mu \circ \phi_1^{-1} : \mathbb{S}^2 \to \mathbb{C}$.

Goal: Construct $\tilde{g} : \underset{\overline{\mathbb{C}}}{\underset{\text{\rotatebox{90}{$\cong$}}}{\mathbb{S}^2}} \to \underset{\overline{\mathbb{C}}}{\underset{\text{\rotatebox{90}{$\cong$}}}{\mathbb{S}^2}} \ni$ B.C. of $\tilde{g} = \tilde{\mu}$

(under "north pole" stereographic proj. as conformal chart)

Identify $\mathbb{S}^2$ with $\overline{\mathbb{C}}$. Define: $\tilde{\mu}_k = k\tilde{\mu}/N$, $k = 0, 1, 2, \ldots N$

Let $\tilde{f}^{\tilde{\mu}_k}$ = Q.C. map associated with $\tilde{\mu}_k$.

$\therefore \tilde{f}^{\tilde{\mu}_0} = \mathrm{Id}$ (assuming $0, 1, \infty$ are fixed)

Proceed to construct:

$$
\begin{tikzcd}
\tilde{\mu}_0 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_1 \arrow[r] \arrow[d, leftrightarrow] & \tilde{\mu}_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_k \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & \tilde{\mu}_N = \tilde{\mu} \arrow[d, leftrightarrow] \\
\underset{= \mathrm{Id}}{\tilde{f}^{\tilde{\mu}_0}} \arrow[r] & \tilde{f}^{\tilde{\mu}_1} \arrow[r] & \tilde{f}^{\tilde{\mu}_2} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_k} \arrow[r] & \cdots \arrow[r] & \tilde{f}^{\tilde{\mu}_N} = \tilde{g}
\end{tikzcd}
$$

Note that the above diagram can be realized iteratively by:

$$
\begin{cases}
\tilde{f}^{\tilde{\mu}_0} = \mathrm{Id} \\
\tilde{f}^{\tilde{\mu}_{k+1}} = \tilde{f}^{\tilde{\mu}_k} + \vec{V}\left(\tilde{f}^{\tilde{\mu}_k}, \frac{\tilde{\mu}}{N}\right),
\end{cases}
$$

where
$$
\vec{V}(f^{\mu}, \nu) = -\int_{\mathbb{C}} \frac{f^{\mu}(w)\left(f^{\mu}(w)-1\right)}{\pi} \left(\frac{\nu(z)\left((f^{\mu})_z(z)\right)^2}{f^{\mu}(z)\left(f^{\mu}(z)-1\right)\left(f^{\mu}(z)-f^{\mu}(w)\right)} \right) dx\, dy
$$

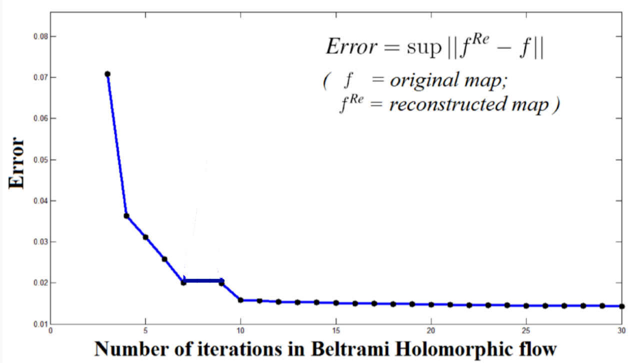
In discrete case, $\vec{V}(f^{\mu}, \nu)$ can be discretized as: $\sum_{v} k(v, w) \overset{\text{(area)}}{A_v}$

Algorithm *Reconstruction of Surface Diffeomorphisms from BCs*

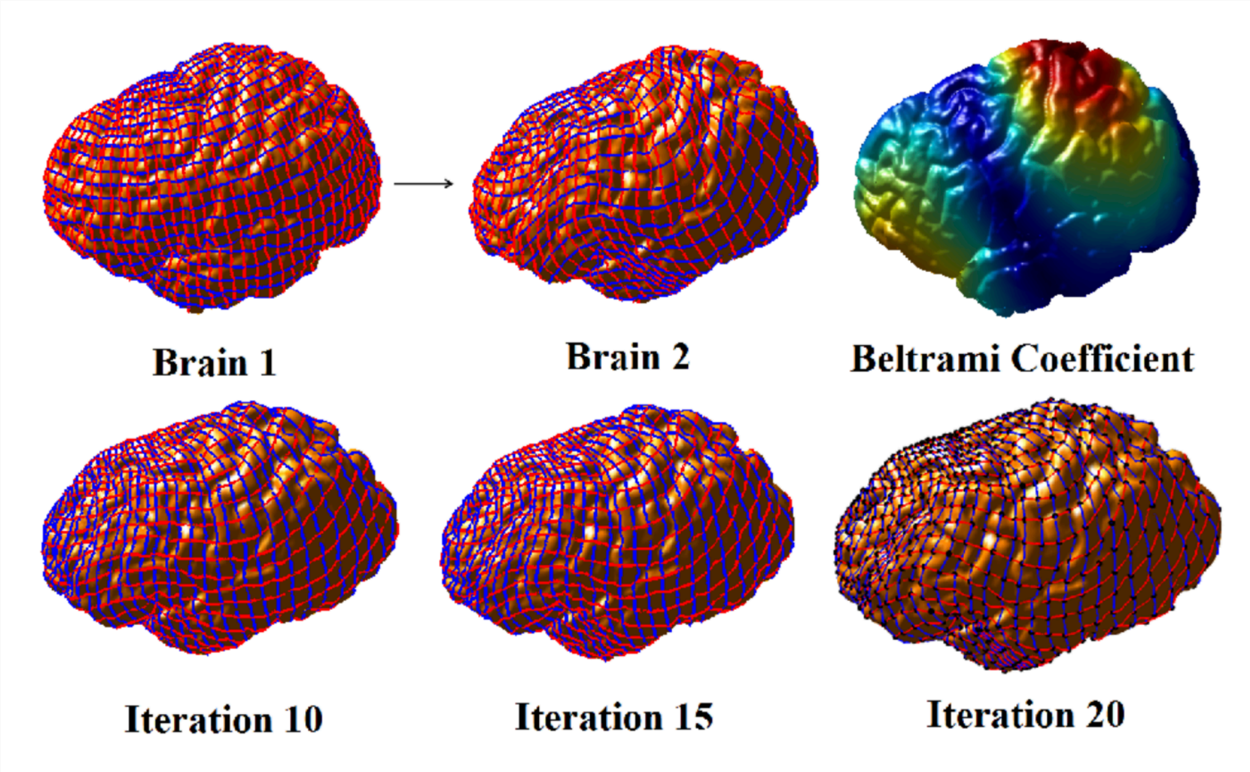
Input: Beltrami Coefficient μ on S_1; conformal parameterizations of S_1 and S_2: ϕ_1 and ϕ_2; Number of iterations N

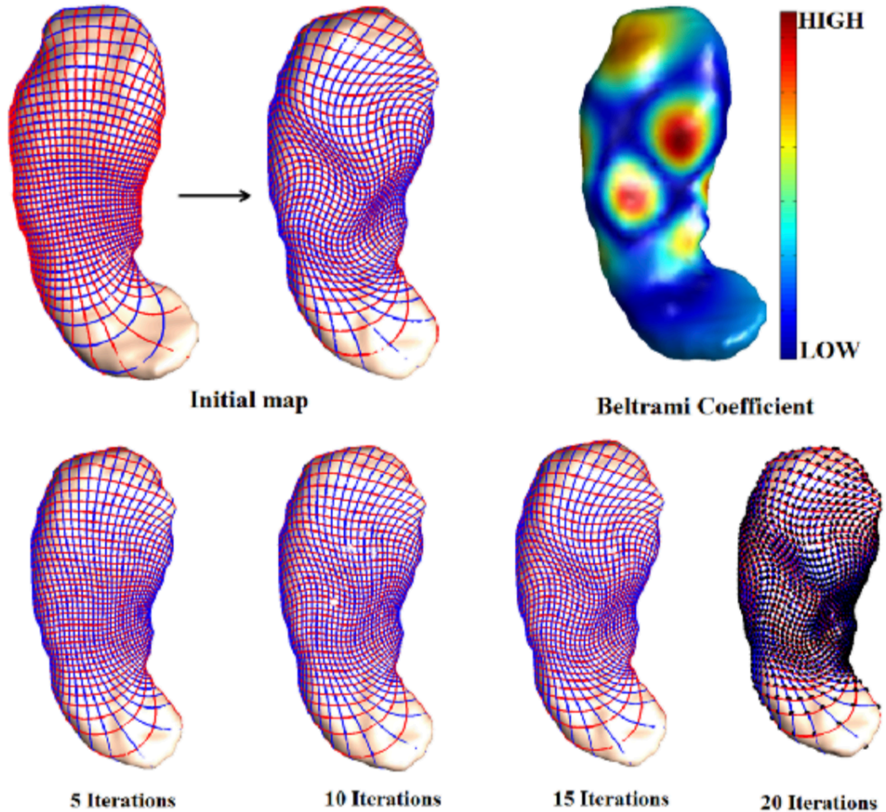
Output: Surface diffeomorphism $f^{\mu}: S_1 \to S_2$ associated to μ.

1) Set $k = 0$; $\widetilde{f}^{\widetilde{\mu}_0} = \mathbf{Id}$.
2) Set $\widetilde{\mu}_k := k\widetilde{\mu}/N$; Compute $\widetilde{f}^{\widetilde{\mu}_{k+1}} = \widetilde{f}^{\widetilde{\mu}_k} + V(\widetilde{f}^{\widetilde{\mu}_k}, \frac{\widetilde{\mu}}{N})$; $k = k + 1$.
3) Repeat Step 2 until $k = N$; Set $f^{\mu} := \phi_2^{-1} \circ \widetilde{f}^{\widetilde{\mu}} \circ \phi_1: S_1 \to S_2$.



Remark: Can be used to solve optimization problem of mappings represented by Beltrami coefficients.


Brain 1
Brain 2
Beltrami Coefficient
Iteration 10
Iteration 15
Iteration 20


HIGH
LOW
Initial map
Beltrami Coefficient
5 Iterations
10 Iterations
15 Iterations
20 Iterations

Example: Let $\phi_1 : S_1 \to D$ and $\phi_2 : S_2 \to D$ be global conformal parameterizations of S_1 and S_2 respectively.

Want to find $f : D \to D$ s.t. f minimizes:

$$E(f) = \int_D (F_1(w) - F_2 \circ f(w))^2 + |\mu(f)(w)|^2 \, dw$$

It can be formulated in term of μ:

$$\tilde{E}(\mu) = \int_D \left(F_1(w) - F_2 \circ f^{\mu}(w)\right)^2 + |\mu(w)|^2 \, dw.$$

To iteratively minimize $\tilde{E}$, consider:

$$\frac{d}{dt}\Big|_{t=0} \tilde{E}(\mu + t\nu) = \int_D \frac{d}{dt}\Big|_{t=0} (F_1(w) - F_2 \circ f^{\mu+t\nu}(w))^2 + |\mu(w) + t\nu(w)|^2 dw$$

$$= -\left(\int_D 2(F_1 - F_2(f^{\mu}))\nabla F_2(f^{\mu}) \frac{d}{dt}\Big|_{t=0} f^{\mu+t\nu} - 2\mu \cdot \nu\right)$$

Let $\begin{pmatrix} A \\ B \end{pmatrix} = 2\left(F_1 - F_2(f^\mu)\right) \nabla F_2$; $\mu = \mu_1 + i\mu_2 \equiv \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$; $\nu = \nu_1 + i\nu_2 \equiv \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$.

Also, we can write: $\vec{V}(f^\mu, \nu) \left(= \frac{d}{dt}\Big|_{t=0} f^{\mu + t\nu}\right) = \int_D \begin{pmatrix} G_1\nu_1 + G_2\nu_2 \\ G_3\nu_1 + G_4\nu_2 \end{pmatrix} dz$

Then: $$\frac{d}{dt}\Big|_{t=0} E(\mu + t\nu) = -\int_D \left[\int_D \begin{pmatrix} A \\ B \end{pmatrix} \cdot \begin{pmatrix} G_1\nu_1 + G_2\nu_2 \\ G_3\nu_1 + G_4\nu_4 \end{pmatrix} dz - 2\mu\cdot\nu \right] dw$$

$$= -\int_D \left(\int_D \begin{pmatrix} AG_1 + BG_3 \\ AG_2 + BG_4 \end{pmatrix} dw - 2\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right) \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} dz$$

$\therefore$ The descent direction for μ_1 is given by:

$$\frac{d\mu_1}{dt} = \int_D (AG_1 + BG_3)\, dw - 2\mu_1 \quad \text{and}$$

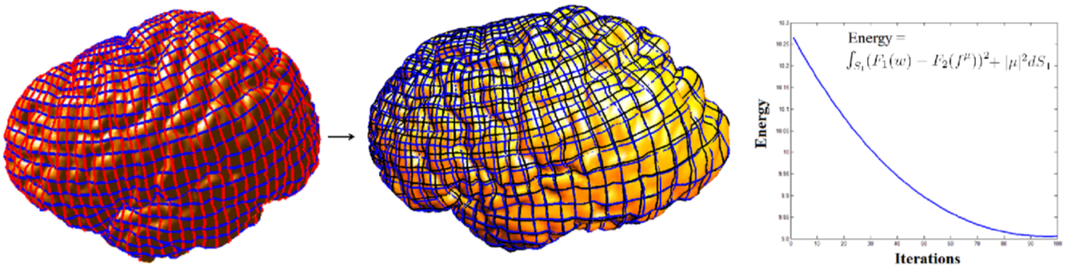
$$\frac{d\mu_2}{dt} = \int_D (AG_2 + BG_4)\, dw - 2\mu_2$$

$\therefore$ We can iteratively optimize the energy E as follows:

$$\mu^{n+1} = \mu^n + dt \underbrace{\begin{pmatrix} \int_D \left(A^n G_1^n + B^n G_3^n\right) dw - 2\mu_1^n \\ \int_D \left(A^n G_2^n + B^n G_4^n\right) dw - 2\mu_2^n \end{pmatrix}}_{\nu^n}$$

$$\begin{tikzcd} \mu^n \arrow[r] \arrow[d, leftrightarrow] & \mu^{n+1} \arrow[d, leftrightarrow] \\ f^{\mu^n} & f^{\mu^{n+1}} = f^{\mu^n} + \vec{V}\left(f^{\mu^n}, \nu^n\right) \end{tikzcd}$$

$\therefore$ We get (μ^n, f^{μ^n}) iteratively.



(A) Brain 1 (B) Brain 2 (C) Energy in each iteration

Fig. 7

ILLUSTRATION OF BHF OPTIMIZATION SCHEME ON BRAIN SURFACES. THIS EXAMPLE SHOWS THE OPTIMIZATION RESULT OF ATCHING TWO FEATURE FUNCTIONS F_1 AND F_2 ON THE TWO BRAIN SURFACES. THE BLUE GRID REPRESENTS THE INITIAL MAP, WHILE THE BLACK GRID REPRESENTS THE OPTIMIZED MAP.

with $F_1 \stackrel{\text{def}}{=} 5.2x^2 + 3.3y^2$

$F_2 \stackrel{\text{def}}{=} 6.8x^2 + 2.8y$

Theorem 4.2 (Beltrami holomorphic flow on $\mathbb{D}$) *There is a one-to-one correspondence between the set of quasiconformal diffeomorphisms of* $\mathbb{D}$ *that fix the points* 0 *and* 1 *and the set of smooth complex-valued functions* μ *on* $\mathbb{D}$ *for which* $\|\mu\|_\infty = k < 1$. *Furthermore, the solution* f^μ *depends holomorphically on* μ. *Let* $\{\mu(t)\}$ *be a family of Beltrami coefficients depending on a real or complex parameter* t. *Suppose also that* $\mu(t)$ *can be written in the form*

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\epsilon(t)(z)$$

for $z \in \mathbb{D}$, *with suitable* μ *in the unit ball of* $C^\infty(\mathbb{D})$, $\nu, \epsilon(t) \in L^\infty(\mathbb{D})$ *such that* $\|\epsilon(t)\|_\infty \to 0$ *as* $t \to 0$. *Then for all* $w \in \mathbb{D}$

$$f^{\mu(t)}(w) = f^\mu(w) + tV(f^\mu, \nu)(w) + o(|t|)$$

locally uniformly on $\mathbb{D}$ *as* $t \to 0$, *where*

$$V(f^\mu, \nu)(w) = -\frac{f^\mu(w)(f^\mu(w)-1)}{\pi}\Bigg(\int_{\mathbb{D}} \frac{\nu(z)((f^\mu)_z(z))^2}{f^\mu(z)(f^\mu(z)-1)(f^\mu(z)-f^\mu(w))}\,dx\,dy$$
$$+\int_{\mathbb{D}} \frac{\overline{\nu(z)}\,\overline{((f^\mu)_z(z))}^2}{\overline{f^\mu(z)}(1-\overline{f^\mu(z)})(1-\overline{f^\mu(z)}f^\mu(w))}\,dx\,dy.\Bigg).$$

Lemma 7.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be a diffeomorphism of the unit disk fixing 0 and 1 and satisfying the Beltrami equation $f_{\bar{z}} = \mu f_z$ with μ defined on $\mathbb{D}$. Let $\tilde{f}$ be the extension of f to $\overline{\mathbb{C}}$ defined as*

$$\tilde{f}(z) = \begin{cases} f(z), & \text{if } |z| \leq 1, \\ \frac{1}{\overline{f(1/\bar{z})}}, & \text{if } |z| > 1. \end{cases}$$

Then $\tilde{f}$ satisfies the Beltrami equation

$$\tilde{f}_{\bar{z}} = \tilde{\mu} \tilde{f}_z$$

on $\overline{\mathbb{C}}$, where the Beltrami coefficient $\tilde{\mu}$ is defined as

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & \text{if } |z| \leq 1, \\ \frac{z^2}{\bar{z}^2} \overline{\mu(1/\bar{z})}, & \text{if } |z| > 1. \end{cases}$$