Lecture 5:

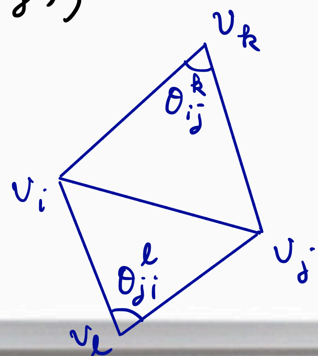
Computation of discrete harmonic map

Let M be a triangulated surface. A piecewise linear function or map is a function/map on M such that it is linear on each triangular face.

Theorem: Given a piecewise linear function $f: M \to \mathbb{R}$, then the harmonic energy of f is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2$$

where



$$w_{ij} = \cot \theta_{ij}^k + \cot \theta_{ji}^l$$

(Cotangent formula)

Definition: (Bary-centric coordinates)

Given a Euclidean triangle with v_i, v_j, v_k, the bary-centric coordinates of a planar point $p \in \mathbb{R}^2$ with respect to the triangle are $(\lambda_i, \lambda_j, \lambda_k)$, $p = \lambda_i v_i + \lambda_j v_j + \lambda_k v_k$, where

$$\lambda_i = \frac{(v_j - p) \times (v_k - v_j) \cdot \vec{n}}{(v_j - v_i) \times (v_k - v_i) \cdot \vec{n}}$$

λ_j, λ_k are defined similarly.

Remark:

- $\lambda_i + \lambda_j + \lambda_k = 1$ (Check)
- If p is the interior point of the triangle, then all components of the bary-centric coordinates are positive.

Lemma: Suppose $f: \Delta^{\subseteq \mathbb{R}^2} \to \mathbb{R}$ is a linear function,

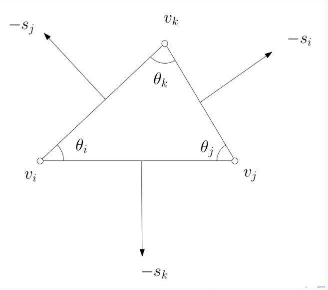
$$f(p) = \lambda_i f(v_i) + \lambda_j f(v_j) + \lambda_k f(v_k),$$

then the gradient of f is: (A = area of Δ)

$$\nabla f(p) = \frac{1}{2A}\left(s_i f(v_i) + s_j f(v_j) + s_k f(v_k)\right).$$

Thus, the harmonic energy on a triangle Δ is given by:

$$\int_\Delta \langle \nabla f, \nabla f \rangle \, dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2$$



Proof: Note that:

$$S_i + S_j + S_k = n \times \{(v_k - v_j) + (v_i - v_k) + (v_j - v_i)\} = \vec{0}$$

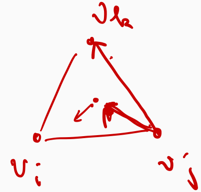
$$\therefore \langle S_i, S_i \rangle = \langle S_i, -S_j - S_k \rangle = -\langle S_i, S_j \rangle - \langle S_i, S_k \rangle.$$

Pick a point $p = \lambda_i v_i + \lambda_j v_j + \lambda_k v_k$. The bary-centric coordinates are given by:

$$\lambda_i = \frac{1}{2A} \langle \underbrace{(v_k - v_j) \times (p - v_j)}_{|v_k - v_j|\,|p - v_j| \sin\theta}, \vec{n} \rangle = \frac{1}{2A} \langle \underbrace{\vec{n} \times (v_k - v_j), p - v_j}_{|v_k - v_j|\,|p - v_j| \cos(90 - \theta)} \rangle$$

$$\therefore \lambda_i = \frac{1}{2A} \langle p - v_j, S_i \rangle, \quad \lambda_j = \frac{1}{2A} \langle p - v_k, S_j \rangle$$

$$\lambda_k = \frac{1}{2A} \langle p - v_i, S_k \rangle$$



where A is the triangle area.

$$\therefore \quad f(p) = \lambda_i f_i + \lambda_j f_j + \lambda_k f_k$$

$$= \frac{1}{2A} \langle p - v_j, f_i s_i \rangle + \frac{1}{2A} \langle p - v_k, f_j s_j \rangle + \frac{1}{2A} \langle p - v_i, f_k s_k \rangle$$

$$= \langle p, \frac{1}{2A} (f_i s_i + f_j s_j + f_k s_k) \rangle - \frac{1}{2A} \big(\langle v_j, f_i s_i \rangle + \langle v_k, f_j s_j \rangle + \langle v_i, f_k s_k \rangle \big)$$

Hence, $\quad \nabla f = \frac{1}{2A} (f_i s_i + f_j s_j + f_k s_k)$

$$\therefore \int_\Delta \langle \nabla f, \nabla f \rangle \, dA = \frac{1}{4A} \langle f_i s_i + f_j s_j + f_k s_k, \; f_i s_i + f_j s_j + f_k s_k \rangle$$

(Using the fact that $\langle s_i, s_i \rangle = - \langle s_i, s_j + s_k \rangle$ etc, we can obtain:)

$$= -\frac{1}{4A} \Big(\langle s_i, s_j \rangle (f_i - f_j)^2 + \langle s_j, s_k \rangle (f_j - f_k)^2 + \langle s_k, s_i \rangle (f_k - f_i)^2 \Big)$$

$\because$ $\dfrac{\langle s_i, s_j \rangle}{2A} = -\cot\theta_k$, $\dfrac{\langle s_j, s_k \rangle}{2A} = -\cot\theta_i$, $\dfrac{\langle s_k, s_i \rangle}{2A} = -\cot\theta_j$

$$\therefore \int_{\Delta} \langle \nabla f, \nabla f \rangle \, dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2$$

__Remark:__ • Let $f: M \to \Omega \subseteq \mathbb{R}^2$ be a discrete map between triangular meshes. Then, each triangle $\Delta \subset M$ can be flatten in $\mathbb{R}^2$.

The harmonic energy on each triangle = harmonic energy from flatten triangle to Ω.

• Adding the harmonic energies on all faces together, and merge items associated with the same edge, then each edge contributes $\frac{1}{2} w_{ij} (f_j - f_i)^2$ where

$$w_{ij} = \cot\theta_{ij}^{k} + \cot\theta_{ji}^{l}$$

<u>Definition</u>: (Laplace operator) The discrete Laplacian Δ_{PL} on a piecewise linear function f is

$$\Delta_{PL} f(v_i) = \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)$$

Hence, if f minimizes the discrete harmonic energy, then:

$$\Delta_{PL} f \equiv 0$$

<u>Remark</u>: The motivation of this definition is by taking the derivative of the discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)^2$$

Recall: The Euler-Lagrange eq^t^ of $\int_M |\nabla f|^2$ is given by

$$\Delta f = 0.$$

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

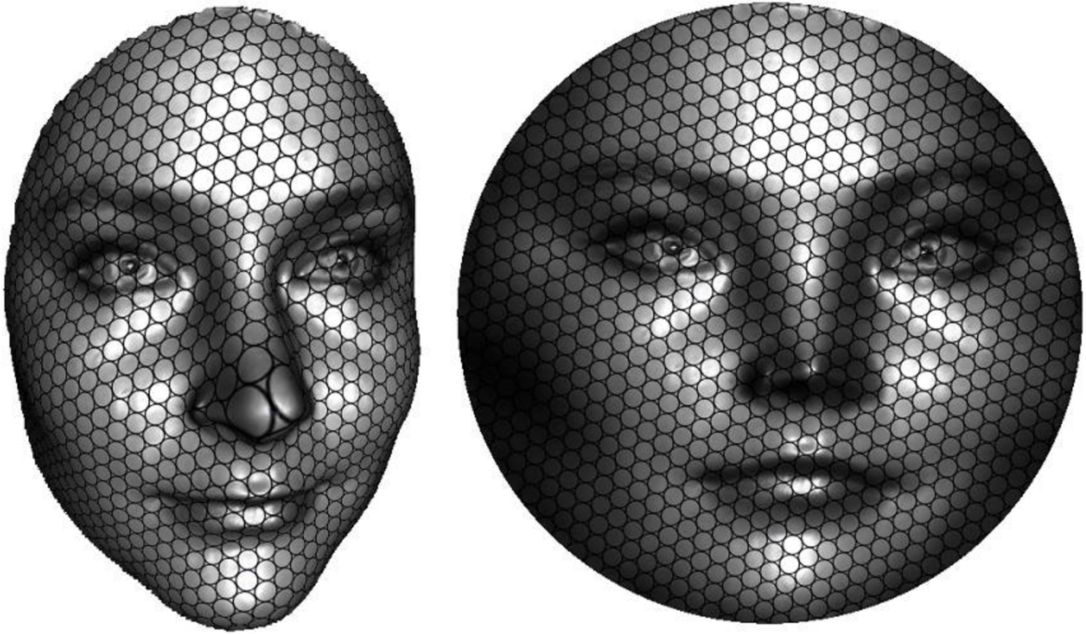


Figure: Harmonic map between topological disks.

Harmonic map v.s. genus-0 surface conformal map

Theorem: (Rado) Let $f: M \to N$ be a harmonic homeomorphism between genus-0 closed surfaces M and N. Then: f is a conformal map.

(Genus-0 : Harmonic $\Longleftrightarrow$ conformal)

Remark: Allow us to compute spherical conformal parameterization by energy minimization (computable!)

Computation of genus-0 spherical conformal parameterization

Let $M = (V, E, F)$ be a triangulation mesh, which is of genus-0.

The harmonic energy of a discrete map $f : M \to \mathbb{S}^2$ (unit sphere) is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)^2$$

(Note that $\|f(v_i)\|^2 = 1$ for $\forall v_i \in V$)

We proceed to minimize $E(f)$ according to a nonlinear heat diffusion process:

$$\frac{d\vec{f}}{dt} = -\mathcal{D}\vec{f} \qquad \left(\vec{f} = \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix} \in \mathbb{R}^{n \times 3} ; \quad \mathcal{D}\vec{f} = \text{descent direction} \right)$$

Definition: The normal component of the Laplacian is:

$$(\Delta f(v_i))^{\perp} = \langle \Delta f(v_i), \vec{n}(f(v_i)) \rangle \vec{n}(f(v_i))$$

↑ unit normal direction at $f(v)$

$$\Delta f(v) = \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right) \in \mathbb{R}^3$$

The tangential component of the Laplacian is:

$$(\Delta f(v_i))^{\parallel} = \Delta f(v_i) - (\Delta f(v_i))^{\perp}.$$

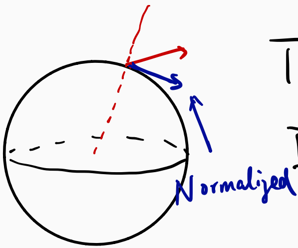
According to gradient descent algorithm, the non-linear diffusion eqt is given by:

$$\frac{d \vec{f}(v,t)}{dt} = -(\Delta f(v))^{\parallel}$$

Remark:

- $E(\vec{f})$ is minimized over $\vec{f}: V \to \mathbb{S}^2$ or $\vec{f} \in M_{N\times 3}$, where $N = \#$ of vertices.

  The descent direction $\vec{d}$ is in $\mathbb{R}^3$.



  If we descend $E(\vec{f})$ along the descent direction without normalization, $\vec{f} + \Delta t \vec{d}$ may go outside $\mathbb{S}^2$.

  $\therefore$ Normalize $\vec{d}$ to the tangential direction on $\mathbb{S}^2$

- $\vec{f} + \Delta t \vec{d}$ may not lie perfectly on $\mathbb{S}^2$, we need to normalize again:

$$\frac{\vec{f} + \Delta t \vec{d}}{\|\vec{f} + \Delta t \vec{d}\|}$$

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;
Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$

3. project the Laplacian to the tangent plane,

normal of the sphere at $\varphi(v_i)$

$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i) \rangle \varphi(v_i)$$

4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;

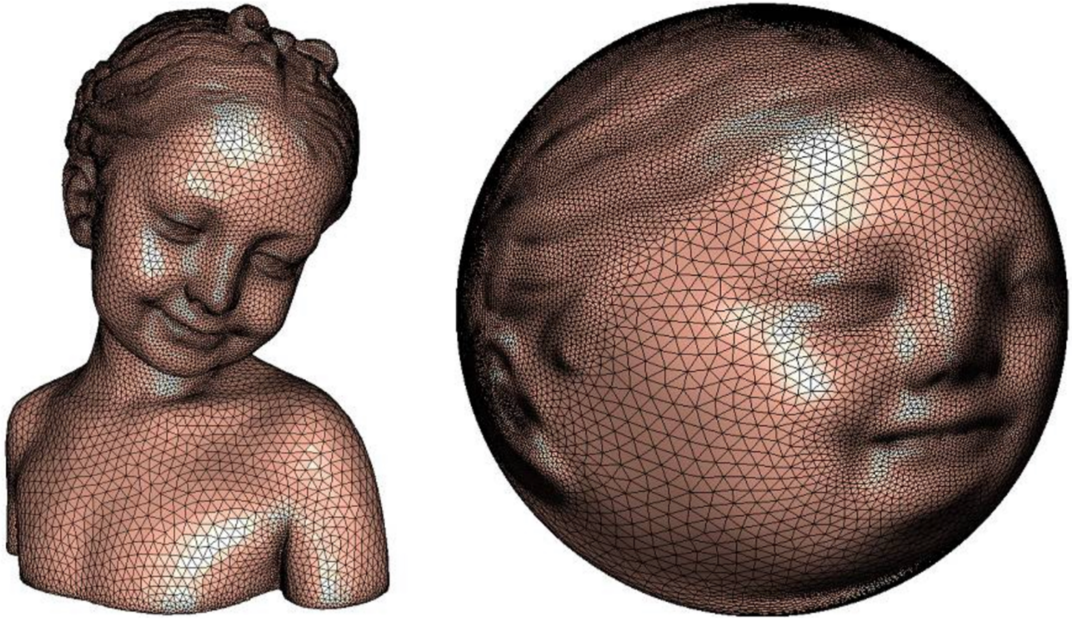
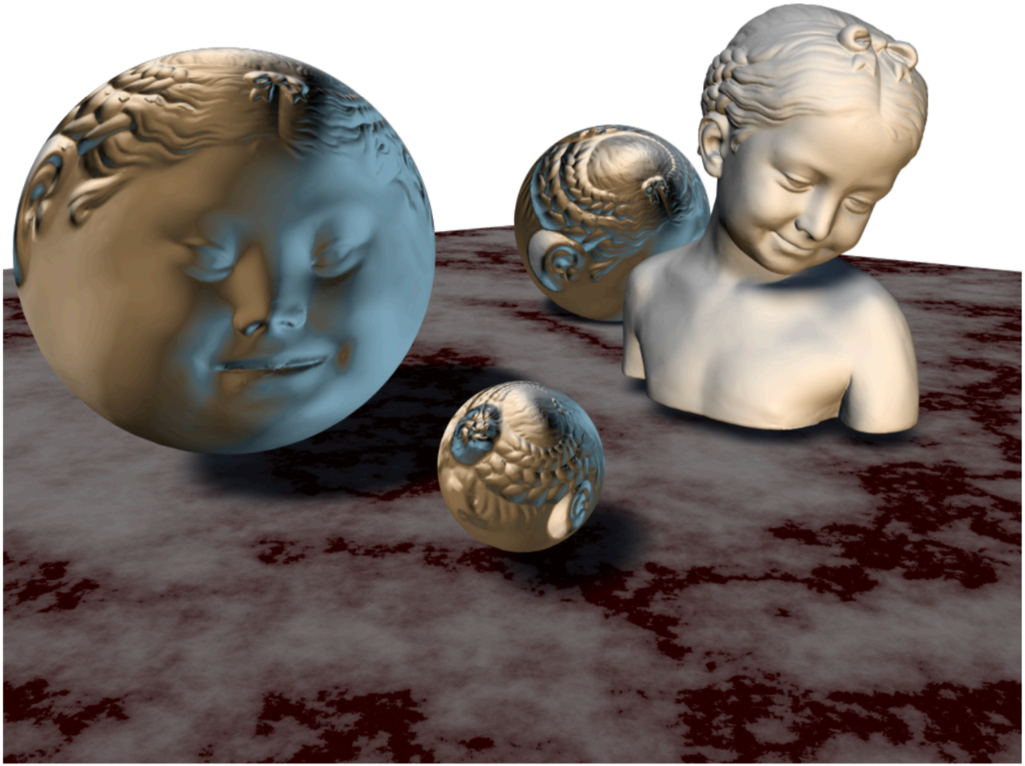


Figure: Harmonic map between topological spheres.



Computation of disk conformal parameterization

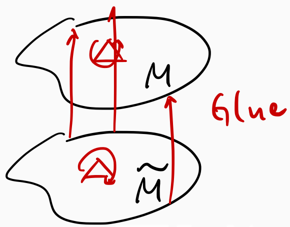
Goal: Given a simply-connected surface S, find $\phi : S \to \mathbb{D}$

Challenge: Cannot get it by computing harmonic map.

(Given a boundary homeomorphism $h : \partial S \to \partial \mathbb{D}$, there exist a unique harmonic map $H : S \to \mathbb{D} \ni H|_{\partial S} = h$. Amongst them, only few of them are conformal w/ suitable boundary conditions)

Idea: (Double covering)

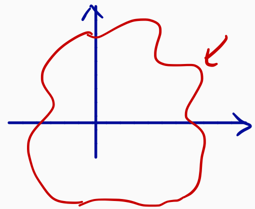
- Let $M = (V, E, F)$, construct $\tilde{M} = (V, \tilde{E}, \tilde{F})$ which is a "reflected" copy of M ($\tilde{F}$ has opposite orientation as F).
- Glue them along the boundary.
- $M \cup \tilde{M} = \bar{M}$ becomes a genus-0 closed surface



- $\tilde{M}$ can be parameterized onto $\mathbb{S}^2$ using the previous algorithm.

<u>Problem</u>: $\bar{M} \xrightarrow{f}$

$\xrightarrow{\phi}$



$f(\partial M)$ may not be on the equator.

$\because$ $\bar{M}$ is a symmetric surface, $\exists$ a conformal map $h : \bar{M} \to \bar{\mathbb{C}}$

such that $h(\partial M) = \partial \mathbb{D}$.

<u>Solution</u>: Picking v_0, v_1, v_2 on ∂M. Reparameterize $\phi \circ f$ by a Mobius Transformation τ such that $h = \tau \circ \phi \circ f$ maps v_0, v_1, v_2 to $0, 1, i$ respectively.

e.g. $\underset{\mathbb{C}}{\underset{\cap}{z_0}}, \underset{\mathbb{C}}{\underset{\cap}{z_1}}, \underset{\mathbb{C}}{\underset{\cap}{z_2}}$ to $0, 1, \infty$ can be done by: $\tau_1 = \dfrac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)}$

etc ...

Input: A oriented surface with boundaries M;
Output: The double covering $\bar{M}$;

1. Make a copy of M, denoted as M';
2. Reverse the order of the vertices of each face of M';
3. Glue M and M' along their corresponding boundary edges to obtain $\bar{M}$.