Lecture 4:

Recap: Let M be a smooth surface.

- A Riemannian metric g associated to M is defined:

For $\forall p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ defines an inner product $\ni$ $\underbrace{\langle \vec{v}, \vec{w} \rangle}_{\text{inner product}} = g_p(\vec{v}, \vec{w})$ for all $\vec{v}, w \in T_pM$

(From Linear Algebra, at each $p \in M$, g_p is associated to a 2×2 SPD matrix $\begin{pmatrix} g_{11}(p) & g_{12}(p) \\ g_{21}(p) & g_{22}(p) \end{pmatrix}$ and in every smooth local coordinates (x^1, x^2),

$$g_p = \sum_{i,j=1}^{2} g_{ij}(p)\, dx^i\, dx^j.$$

g_{ij}'s are smooth)

- Any metric surface M is associated to an isothermal coordinates.

  That's, let $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ be the conformal atlas for M. ($z_\alpha = x_\alpha + i y_\alpha$)

  Then: $$g = e^{\lambda(z_\alpha)} \left(dx_\alpha^2 + dy_\alpha^2 \right)$$

- Any metric surface is a Riemann surface.

- $$\begin{tikzcd} S_1 \arrow[r, "f"] \arrow[d, "z_\alpha"] & S_2 \arrow[d, "w_\beta"] \\ U_\alpha \arrow[r] & V_\beta \end{tikzcd}$$

  $\tilde{f} = w_\beta \circ f \circ z_\alpha^{-1}$

  f is conformal iff $\tilde{f}$ is conformal for all z_α, w_β

Basic theories of planar conformal maps

Theorem: (Riemann mapping) Suppose $D \subset \mathbb{C}$ is a simply-connected domain on the complex plane, the boundary ∂D has more than one point, $z_0 \in D$ is an arbitrary interior point. Then, there exists a unique conformal mapping $\phi : D \to \Delta$ from D to the unit disk Δ, such that $\phi(z_0) = 0$ and

$\phi'(z_0) > 0$.

Remark: If $f : S \to \mathbb{D}$ and $g : S \to \mathbb{D}$ are disk conformal parameterization of S, then: $g \circ f^{-1}$ is a conformal map between unit disk

$$\begin{array}{ccc} S & & S \\ f\downarrow & & \downarrow g \\ \mathbb{D} & \to & \mathbb{D} \end{array}$$

$\therefore \quad g \circ f^{-1}(z) = \underbrace{e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)}_{\phi(z)}$ for some $a \in \mathbb{C}$, $\theta \in [0, 2\pi)$

$\therefore \quad \boxed{g = f \circ \phi}$

Surface harmonic map : theories and computation

Basic theoretical background

1. Let $f : M \to \mathbb{R}$. The differential of f is defined as:

$$df_p(\vec{v}) \stackrel{\text{def}}{:=} D_{\vec{v}} f \quad \text{for } \forall \vec{v} \in T_pM$$

$$= \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) \quad \text{where} \quad \frac{d}{dt}\Big|_{t=0} \gamma(t) = \vec{v}$$

Under the coordinate chart (x^1, x^2) around p,

$$df_p := \sum_{i=1}^{2} \frac{\partial f}{\partial x^i}(p)\, dx^i$$

2. (Planar harmonic function) Let $\Omega \subseteq \mathbb{R}^2$ and let $u : \Omega \to \mathbb{R}$.

u is said to be a harmonic function if : $\Delta u = 0$

Harmonic map and energy minimization

Consider: $E(u) \overset{\text{def}}{=} \int_\Omega \langle \nabla u, \nabla u \rangle \, dx\,dy$

Suppose u minimizes $E(u)$, then:

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega \langle \nabla u + \varepsilon \nabla h, \nabla u + \varepsilon \nabla h \rangle \, dx\,dy$$

$$= 2 \int_\Omega \langle \nabla u, \nabla h \rangle \, dx\,dy$$

Fixing the boundary, we have $h \equiv 0$ on $\partial\Omega$.

Integration by part gives : $0 = 2\int_\Omega h \Delta u \, dx\,dy$ for $\forall h$, $h|_{\partial\Omega} \equiv 0$

$$\therefore \quad \begin{cases} \Delta u \equiv 0 \\ u|_{\partial\Omega} = g \quad \text{(Boundary condition)} \end{cases}$$

Remark: • A harmonic function minimizes the harmonic energy $E(u) = \int_\Omega \langle \nabla u, \nabla u \rangle \, dx dy$

• A map $f : \Omega \subseteq \mathbb{R}^2 \to \Omega' \subseteq \mathbb{R}^2$ is said to be harmonic if $f \overset{\text{def}}{:=} u + iv$, $\Delta u \equiv 0$ and $\Delta v \equiv 0$.

• A map $f : S \to \Omega \subseteq \mathbb{R}^2$ (where S is a Riemann surface) is a harmonic map if with respect to a (conformal) coordinate chart ϕ, $f \circ \phi$ is a harmonic map.

More concrete definition of harmonic map between Riemann surfaces

Definition: Let $f: M \to \mathbb{R}$ be a smooth function on M.

The Riemannian gradient ∇f_p at $p \in M$ is defined $\ni$

for $\forall p \in M$, $\forall \vec{v} \in T_pM$, ∇f_p is a tangent vector at p

satisfying:
$$\langle \nabla f_p, \vec{v} \rangle_g = D_{\vec{v}} f.$$

In any smooth coordinate (x^1, x^2),

$$\nabla f_p = \sum_{i,j=1}^{2} g^{ij}(p) \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^j}\Big|_p \quad \text{where} \quad (\text{Check})$$

(g^{ij}) is the inverse of (g_{ij}).

<u>Remark:</u> Let $f: S \to \Omega \subseteq \mathbb{R}^2$. f is a harmonic map,

if: f minimizes $E(f) = \int_S \langle \nabla f_p, \nabla f_p \rangle_g$

(Harmonic energy)

<u>How about harmonic energy between Riemann surfaces?</u>

Let $f: (M,g) \to (N,h)$ be a homeomorphism.

Define: $E(f) = \frac{1}{2} \int_M \| df \|_h^2 \, \omega_g$ be the harmonic energy

where:

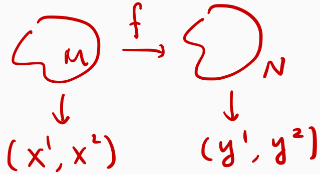
- ω_g is the area measure on M defined by the metric g
- $\| df \|_h^2 \overset{\text{def}}{=} \sum_{i=1}^{2} h_{f(p)} \left(df_p(e_i), df_p(e_i) \right)$,

$\{e_i\}$ is an orthonormal basis on $T_p M$

In local coordinate (x^1, x^2) on M and (y^1, y^2) on N,

$$\|df_p\|_h^2 = \sum_{i,j=1}^{2} \sum_{\alpha,\beta=1}^{2} g^{ij}(p)\, h_{\alpha\beta}(f(p))\, f_i^\alpha f_j^\beta \quad \text{where}$$

$$f_i^\alpha \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i}\left(y^\alpha \circ f\right), \quad f_j^\beta \stackrel{\text{def}}{=} \frac{\partial}{\partial x^j}\left(y^\beta \circ f\right)$$



and $\omega_g = \sqrt{|g|}\, dx^1 dx^2$ and $|g|$ = determinant of (g_{ij})

<u>Definition:</u> The homeomorphism $f : M \to N$ is a harmonic map if f minimizes the harmonic energy.

Important fact:

Theorem: Suppose a harmonic map $\varphi : (S, g) \to \Omega \subseteq \mathbb{R}^2$ satisfies:

① Ω is convex;

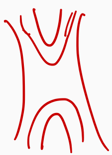
② the restriction of $\varphi : \partial S \to \partial \Omega$ on the boundary is homeomorphic

Then: u is diffeomorphic in the interior of S.

Proof: By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi : (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated at p.

$\therefore \exists\, a, b \in \mathbb{R}$ (not all zero) such that:

$$a \nabla u(p) + b \nabla v(p) = 0$$



By $\Delta u = \Delta v = 0$, the auxilary function

$$f(q) = a\,u(q) + b\,v(q) \text{ is also harmonic.}$$

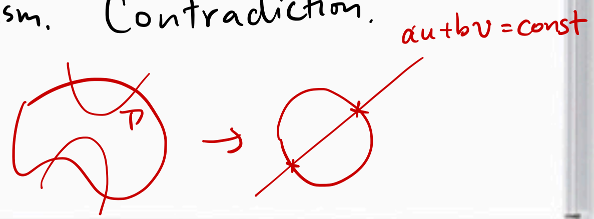
$\because \nabla f(p) = 0 \quad \therefore\ p$ is a saddle point of f.

Consider $\Gamma = \{ q \in S \mid f(q) = f(p) - \varepsilon \}$ (level set of f near p)

Γ has two connected components, intersecting ∂S at 4 points.

But Ω is a planar convex domain, $\partial\Omega$ and the line $au + bv = \text{const}$ have two intersection points. By assumption, $\varphi|_{\partial S}$ is a homeomorphism. Contradiction.

$\therefore\ \varphi$ is homeomorphic.



Theorem: If $f : S \to \underset{\uparrow \text{convex}}{\Omega} \subseteq \mathbb{R}^2$ and $g : S \to \Omega$ are both harmonic map satisfying $f|_{\partial S} = g|_{\partial S} = \underset{(\text{given})}{h}$,

then: $f \equiv g$.

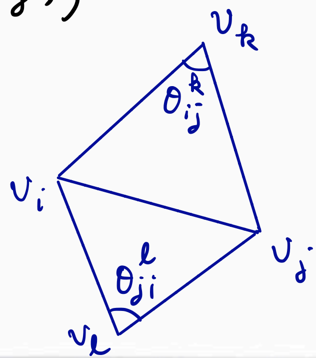
Computation of discrete harmonic map

Let M be a triangulated surface. A piecewise linear function or map is a function/map on M such that it is linear on each triangular face.

Theorem: Given a piecewise linear function $f: M \to \mathbb{R}$, then the harmonic energy of f is given by:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_i) - f(v_j)\right)^2$$

where



$$w_{ij} = \cot \theta_{ij}^{k} + \cot \theta_{ji}^{l}$$

(Cotangent formula)

Definition: (Laplace operator) The discrete Laplacian Δ_{PL} on a piecewise linear function f is

$$\Delta_{PL} f(v_i) = \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)$$

Hence, if f minimizes the discrete harmonic energy, then:

$$\Delta_{PL} f \equiv 0$$

Remark: The motivation of this definition is by taking the derivative of the discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} \left(f(v_j) - f(v_i) \right)^2$$

Recall: The Euler-Lagrange eq't of $\int_M |\nabla f|^2$ is given by $\Delta f = 0$.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;

Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.