Lecture 1: Basic mathematical concept

Brief introduction on:

1. Topological surface
2. Riemannian surface

Topological surface

- interested in the topology / genus of the surface ONLY.
- not equipped with a metric measuring distance

Riemannian surface

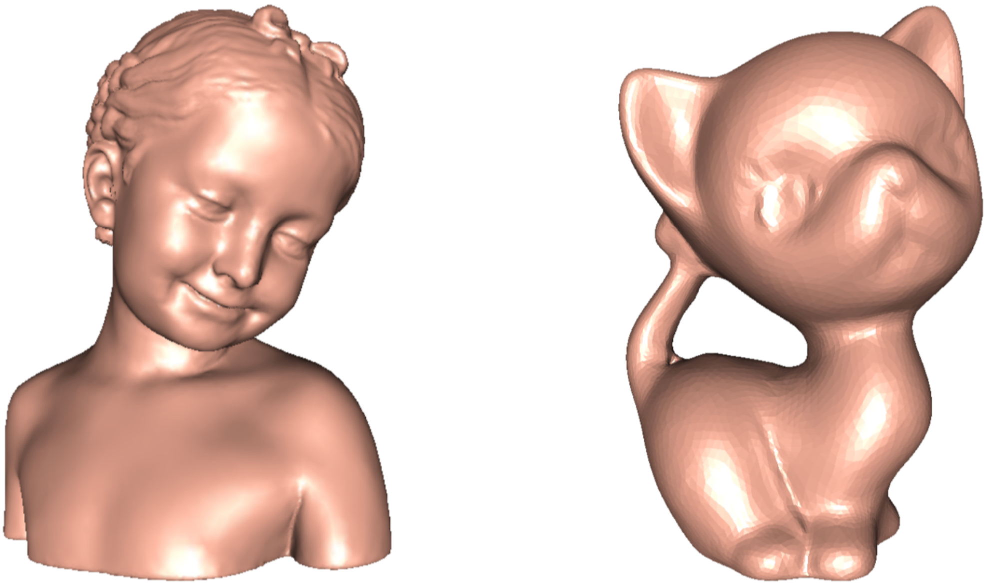
- Equipped with a metric measuring distance
- Topology / genus can be discovered by Gauss-Bonnet Theorem.

$$\int_M K \, dA = 2\pi \chi(M), \quad \chi(M) = 2-2g$$

K: Gaussian curvature

$\chi(M)$: Euler char.

Topological surfaces:



Topological Sphere   Topological Torus

Figure: How to differentiate the above two surfaces.

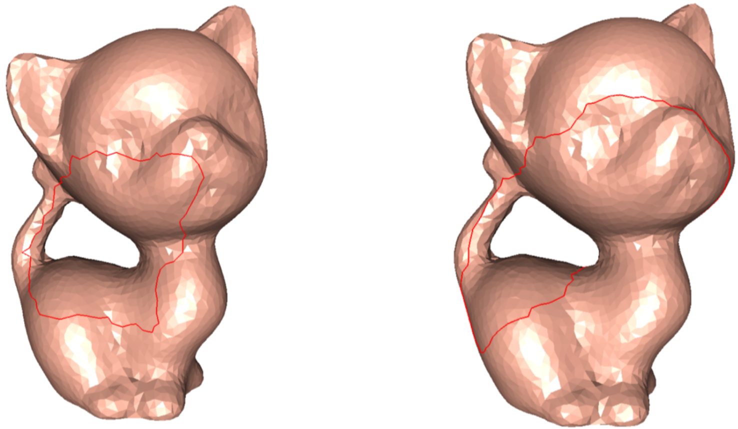


Figure: Check whether all loops on the surface can shrink to a point.

All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use (g, b) to represent the topological type of an oriented surface S.

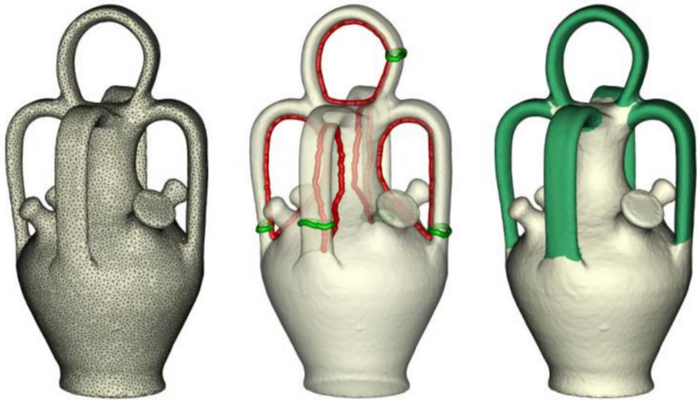


Figure: Handle detection by finding the handle loops and the tunnel loops.

<u>Remark</u>: Topological surface S can be determined by the first homotopy group.

Suppose $q \in S$ is a base point, all oriented loop can be classified by homotopy and hence form a homotopic class.

All homotopic classes form the fundamental group / first homotopic class of S. Denote it by $\pi_1(S, q)$.

Definition: Let $\gamma_1, \gamma_2 : [0,1] \to S$ be two curves. A homotopy connecting γ_1 and γ_2 is a continuous mapping $F: [0,1] \times [0,1] \to S$, such that:

$$F(0,t) = \gamma_1(t) \quad \text{and} \quad F(1,t) = \gamma_2(t).$$

γ_1 is said to be homotopic to γ_2 if there exists a homotopy between them.

Definition: A closed curve (loop) through p is a curve such that $\gamma(0) = \gamma(1) = p$.

Lemma: Homotopy relation is an equivalence relation.

Remark: The homotopy class of a loop γ is denoted by $[\gamma]$.

(If $\gamma_1 \in [\gamma]$, then: γ_1 is homotopic to γ)

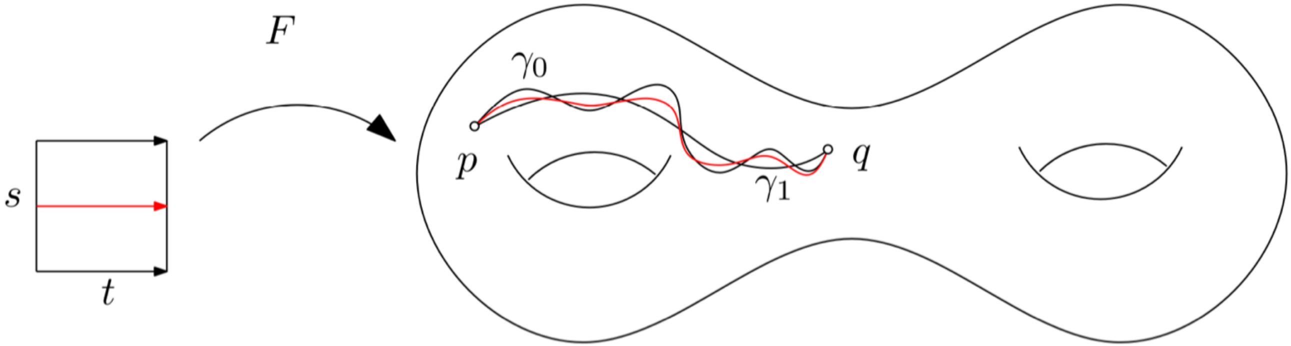


Figure: Path homotopy.

Definition: Let γ_0, γ_1 be two loops through p. The product of two loops is defined as:

$$\gamma_0 \cdot \gamma_1(t) = \begin{cases} \gamma_0(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The loop inverse is defined as:

$$\gamma^{-1}(t) = \gamma(1-t)$$

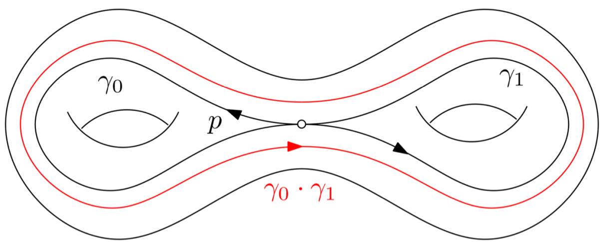


Figure: Loop product

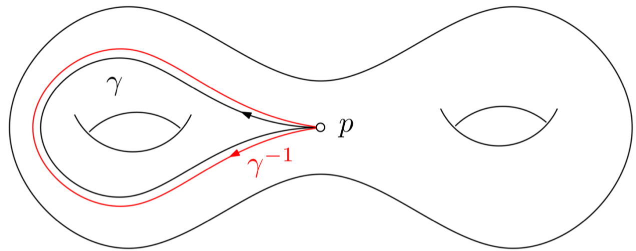
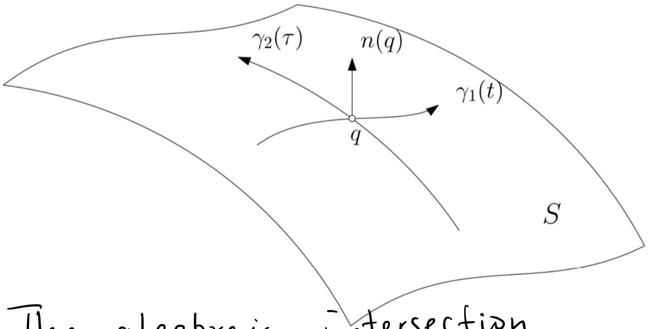


Figure: Loop inversion

Definition: (Intersection index)



Suppose γ_1 and γ_2 intersects at q. That's, $\gamma_1(t) = \gamma_2(\tau) = q$.

Then: the intersection index at q is:

$$\mathrm{Ind}(\gamma_1, \gamma_2, q) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} > 0 \\ -1 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} < 0 \\ 0 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} = 0 \end{cases}$$

The algebraic intersection number of γ_1 and γ_2 is defined as:

$$\gamma_1 \cdot \gamma_2 \overset{\text{def}}{:=} \sum_{q_i \in \gamma_1 \cap \gamma_2} \mathrm{Ind}(\gamma_1, \gamma_2, q_i)$$

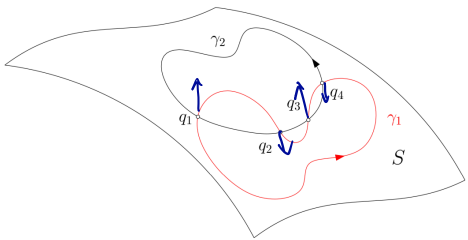


Figure: Algebraic intersection number

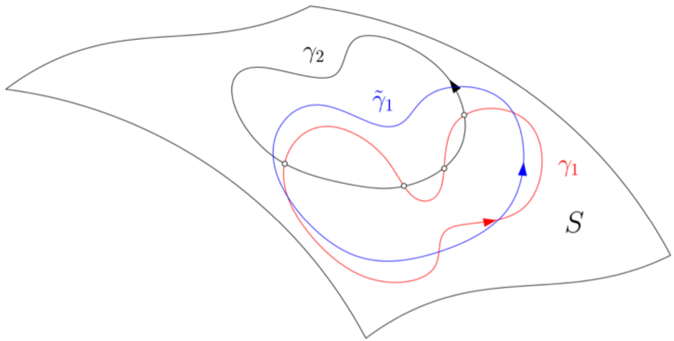


Figure: Algebraic intersection number

Algebraic Intersection Number Homotopy Invariance

Suppose γ_1 is homotopic to $\tilde{\gamma}_1$, then the algebraic intersection number

$$\gamma_1 \cdot \gamma_2 = \tilde{\gamma}_1 \cdot \gamma_2.$$

Proof: Exercise

Definition (Canonical Basis)

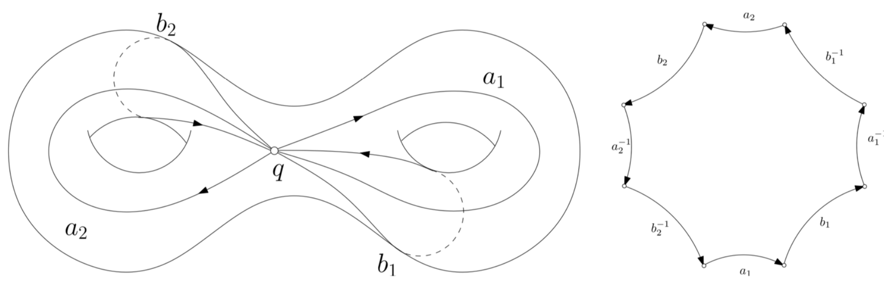
Suppose S is a compact, oriented surface, there exists a set of generators of the fundamental group $\pi_1(S, p)$,

$$G = \{[a_1], [b_1], [a_2], [b_2], \cdots, [a_g], [b_g]\}$$

such that

$$a_i \cdot b_j = \delta_i^j, a_i \cdot a_j = 0, b_i \cdot b_j = 0,$$

where $a_i \cdot b_j$ represents the algebraic intersection number of loops a_i and b_j, δ_{ij} is the Kronecker symbol, then G is called a set of canonical basis of $\pi_1(S, p)$.



Remark: We'll learn how to find a_i's, b_j's $\Rightarrow$ we can cut and flatten!

Universal covering space

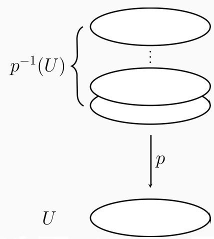
Definition (Covering Space) Let S and $\tilde{S}$ be topological spaces. A continuous map $p : \tilde{S} \to S$ is a <u>covering map</u> if:

(1) For each $q \in S$, $\exists$ neighbourhood U of q such that $p^{-1}(U) = \dot{\bigcup}_i \tilde{U}_i$ is a disjoint union of open sets $\tilde{U}_i$

(2) $p|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a homeomorphism. for $\forall i$.

Then: $\tilde{S}$ is called a <u>covering space</u>.

If $\tilde{S}$ is simply-connected, then $\tilde{S}$ is called a <u>universal covering space</u>.



Definition: (Deck Transformation)

The automorphism of $\tilde{S}$, $\tau: \tilde{S} \to \tilde{S}$, is called a deck transformation if they satisfy $p \circ \tau = p$.

All deck transformations form a group, the covering group, and denoted as $Deck(\tilde{S})$.

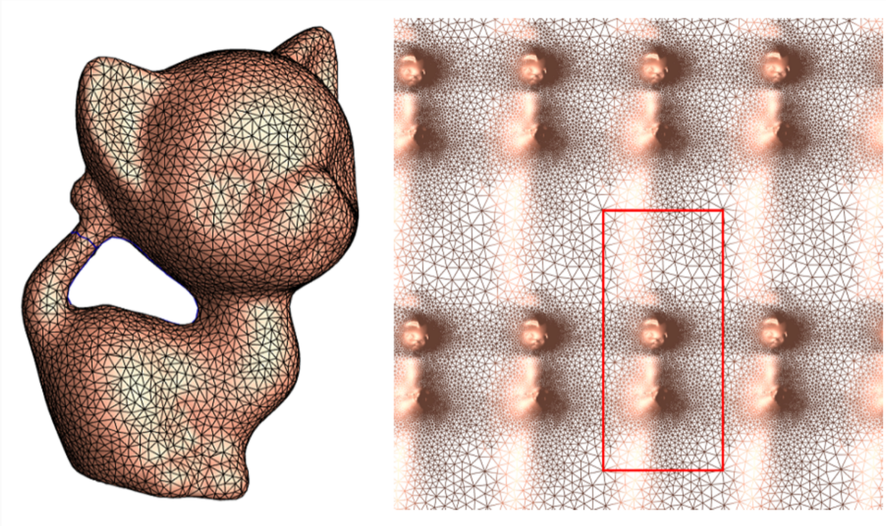
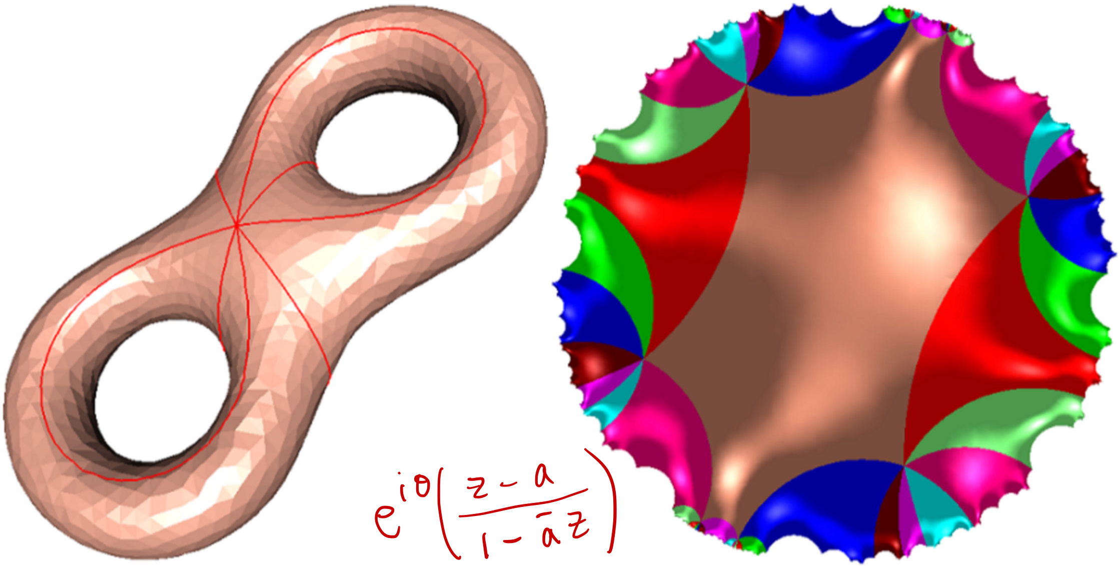


Figure: Universal Covering Space

$Deck(\tilde{S})$

"

Space of translations from one fundamental domain to another.



$$e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)$$

Figure: Universal Covering Space of a genus two surface.

$\mathrm{Deck}(\tilde{S})$ = Space of Möbius transformations.

Smooth manifold

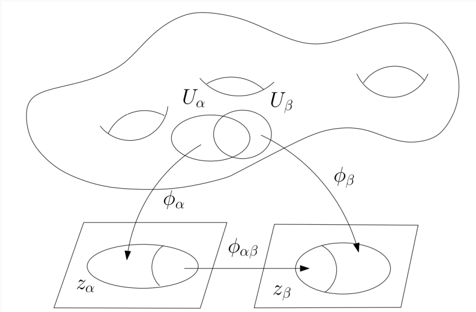
Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.



Definition (Tangent Vector)

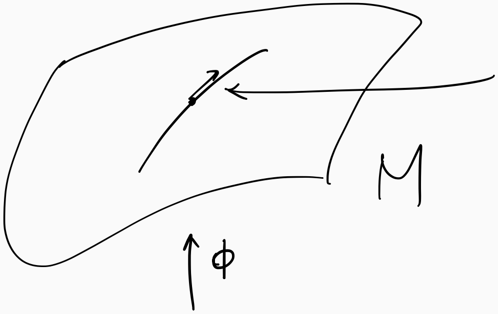
A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

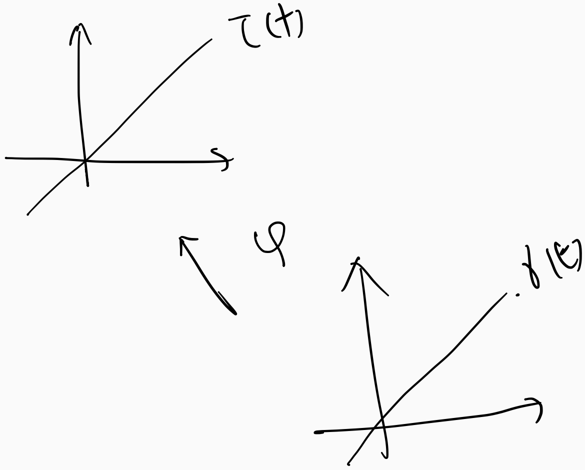
$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

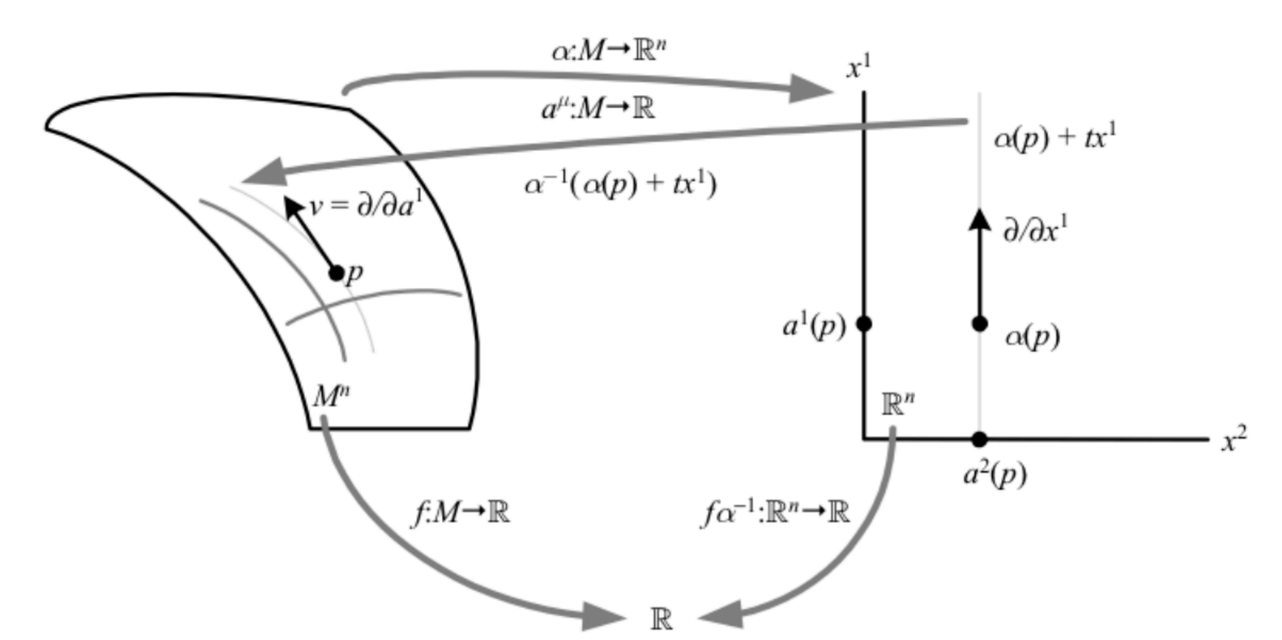
$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.



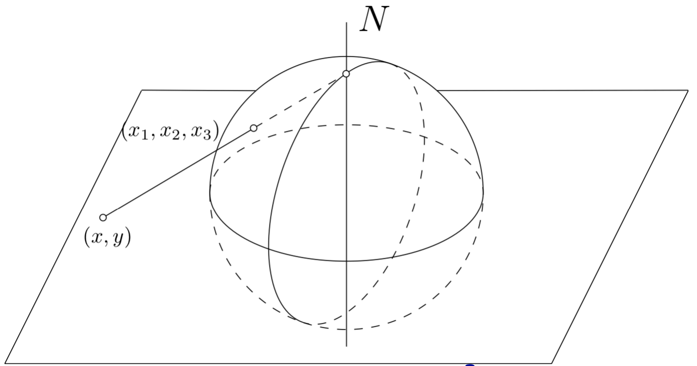
$$\frac{d}{dt}\Big|_{t=0} \phi(\tau(t))$$

$$= \frac{d}{dt}\Big|_{t=0} \phi(\tau \circ \gamma(t))$$





$\alpha{:}M \to \mathbb{R}^n$
$a^{\mu}{:}M \to \mathbb{R}$
x^1
$\alpha(p) + tx^1$
$\alpha^{-1}(\alpha(p) + tx^1)$
$v = \partial/\partial a^1$
p
$\partial/\partial x^1$
$a^1(p)$
$\alpha(p)$
M^n
$\mathbb{R}^n$
x^2
$a^2(p)$
$f{:}M \to \mathbb{R}$
$f\alpha^{-1}{:}\mathbb{R}^n \to \mathbb{R}$
$\mathbb{R}$



Sterographic projection from $\mathbb{C}$ to $\mathbb{S}^2 \setminus \{N\}$.

$\phi : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$ defined by:

$$\phi(x, y) = (x_1, x_2, x_3) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)$$

$$\phi^{-1}(x_1, x_2, x_3) = (x, y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$$

$$\frac{\partial}{\partial x} = \frac{\partial \Phi}{\partial x} = \frac{2}{(1+x^2+y^2)^2}\left(1-x^2+y^2,\ -2xy,\ 2x\right)$$

$$\frac{\partial}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{2}{(1+x^2+y^2)^2}\left(-2xy,\ 1+x^2-y^2,\ 2y\right)$$

Note that:

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = \frac{4}{(1+x^2+y^2)^2}$$

$$\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = \frac{4}{(1+x^2+y^2)^2}$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = 0$$

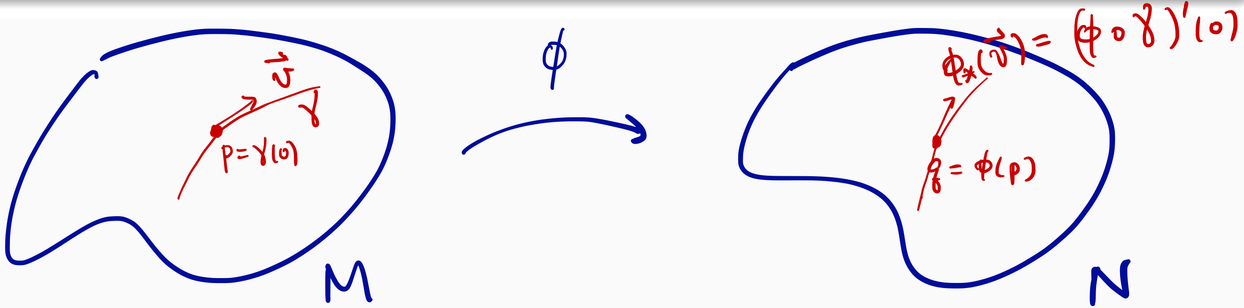
Angle preserving!

Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from M to N, $\gamma : (-\epsilon, \epsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_pM$, then $\phi \circ \gamma$ is a curve on N, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)}N,$$

as the push-forward tangent vector of $\mathbf{v}$ induced by ϕ.



Integration on surface

__Definition__: Suppose $\mathcal{U} \subset M$ is an open set of a 2-dim manifold M, and $\phi : \mathcal{U} \to \Omega \subset \mathbb{R}^2$ is a chart. Then:

$$\int_{\mathcal{U}} f \, dA = \int_{\Omega} f \circ \phi^{-1} \sqrt{EG - F^2} \, du \, dv$$

where $E = (\phi^{-1})_u \cdot (\phi^{-1})_u$, $F = (\phi^{-1})_u \cdot (\phi^{-1})_v$, $G = (\phi^{-1})_v \cdot (\phi^{-1})_v$

__Definition__: Choose a partition of unity $\{\psi_i : \mathcal{U}_i \to \mathbb{R}\}_{i \in I}$ such that $\bigcup_i \mathcal{U}_i = M$, $\psi_i(p) \geq 0$ for $\forall i$ and $\sum_i \psi_i(p) \equiv 1$ for $\forall p \in M$.

Then:
$$\int_M f \, dA = \sum_i \int_{\mathcal{U}_i} \psi_i f \, dA$$
$$= \sum_i \int_{\Omega_i} (\psi_i f) \circ \phi_i^{-1} \sqrt{EG - F^2} \, du \, dv$$

where $\phi_i : \mathcal{U}_i \to \Omega_i$ is a chart.

Gauss-Bonnet Theorem

$\frac{d}{dt}\Big|_{t=0} \vec{N}(\phi(\gamma(t)))$, $\phi(\gamma(0)) = p$, $\frac{d}{dt}\Big|_{t=0} \phi\circ\gamma(t) = \vec{v}$

<u>Definition</u>: Let $p \in M$ // and $\vec{v} \in T_pM$ (tangent plane at p).

Define: $S_p(\vec{v}) = -D_{\vec{v}}\vec{N}$, where $\vec{N}$ is the normal direction of M at p. Then: $S_p : T_pM \to T_pM$ is a linear operator, called the shape operator.

The Gaussian curvature at p is defined as:

$$K = \det(S_p).$$

<u>Theorem</u>: (Gauss-Bonnet) Let M be a compact closed surface.

$$\int_M K \, dA = 2\pi \underbrace{\chi(M)}$$

Euler characteristic

(integer depending on the topology)

Discrete Gauss-Bonnet Theorem

Theorem: For an oriented discrete triangulated surface M,

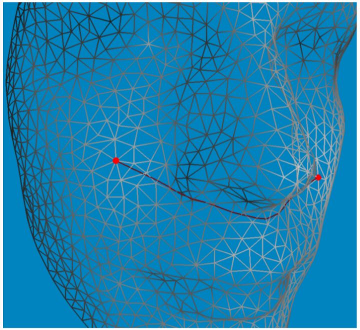
$$\sum_{v_i} K(v_i) = 2\pi \chi(M)$$

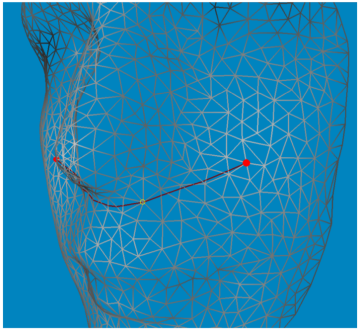
where $\{v_i\}$ is the collection of vertices, $K(\vec{v}_i)$ is the discrete Gaussian curvature defined as:

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases}$$

and $\chi(M) = |V| + |F| - |E|$

(# of vertices, # of faces, # of edges)





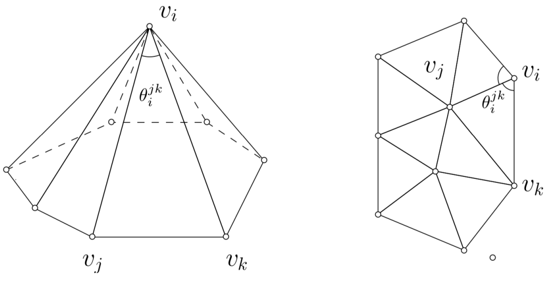


Figure: Discrete Gaussian curvature.