Lecture 9:

Mathematics of JPEG

Consider a $N \times N$ image f. Extend f to a $2M \times 2N$ image $\tilde{f}$, whose indices are taken from $[-M, M-1]$ and $[-N, N-1]$.

Define $f(k,l)$ for $-M \leq k \leq M-1$ and $-N \leq l \leq N-1$ such that

$$f(-k-1, -l-1) = f(k,l) \quad \} \text{ Reflection about } (-1/2, -1/2)$$

$$\left.\begin{aligned} f(-k-1, l) &= f(k,l) \\ f(k, l-1) &= f(k,l) \end{aligned}\right\} \text{ Reflection about the axis } k = -1/2 \text{ and } l = -1/2$$

Example:

$l = -1/2$

$$\begin{array}{cccc|cccc}
 & & & & & & & \\
9 & 8 & 7 & 7 & 8 & 9 & & k=-3 \\
6 & 5 & 4 & 4 & 5 & 6 & & k=-2 \\
3 & 2 & 1 & 1 & 2 & 3 & & k=-1 \\
\hline
3 & 2 & 1 & 1 & 2 & 3 & & k=0 \\
6 & 5 & 4 & 4 & 5 & 6 & & k=1 \\
9 & 8 & 7 & 7 & 8 & 9 & & k=2 \\
l=-3 & l=-2 & l=-1 & l=0 & l=1 & l=2 & &
\end{array}$$

$k = -1/2$

$f(-2,-2)$

$\shortparallel$

$f(1,1)$

Reflection about $(-1/2, -1/2)$.

$f(-1,1)$

$\shortparallel$

$f(0,1)$

Reflection about the axis $k = -1/2$.

Make the extension as a reflection about $(0,0)$, the axis $k=0$ and the axis $l=0$.

Done by shifting the image by $(\frac{1}{2}, \frac{1}{2})$

After shifting

$$
\begin{array}{ccc:ccc|l}
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+(-3) \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+(-2) \\
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+(-1) \\
\hdashline
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+0 \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+1 \\
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+2 \\
\hline
\frac{1}{2}+-3 & \frac{1}{2}+-2 & \frac{1}{2}+-1 & \frac{1}{2}+0 & \frac{1}{2}+1 & \frac{1}{2}+2 &
\end{array}
$$

k

l

Now, we compute the DFT of (shifted) $\tilde{f}$:

$$F(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j\frac{2\pi}{2M}m(k+\frac{1}{2})} e^{-j\frac{2\pi}{2N}n(l+\frac{1}{2})}$$

$$= \frac{1}{4MN} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

$$= \frac{1}{4MN} (\underbrace{\sum_{k=-M}^{-1} \sum_{l=-N}^{-1}}_{A_1} + \underbrace{\sum_{k=-M}^{-1} \sum_{l=0}^{N-1}}_{A_2} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=-N}^{-1}}_{A_3} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=0}^{N-1}}_{A_4})$$

$$f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

After some messy simplication, we can get:

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k + \frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l + \frac{1}{2}\right)\right]$$

Definition: (Even symmetric discrete cosine transform [EDCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M-1$ and $0 \leq l \leq N-1$. The **even symmetric discrete cosine transform (EDCT)** of f is given by:

$$\hat{f}_{ec}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

with $0 \leq m \leq M-1, 0 \leq n \leq N-1$

Remark:
- Smart idea to get a decomposition consisting only of cosine function (by reflection and shifting!)
- Can be formulated in matrix form
- Again, it is a separable image transformation.

- The inverse of EDCT can be explicitly computed. More specifically, the **inverse EDCT** is defined as:

$$f(k,l)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}C(m)C(n)\hat{f}_{ec}(m,n)\cos\frac{\pi m(2k+1)}{2M}\cos\frac{\pi n(2l+1)}{2N} \qquad (**)$$

where $C(0)=1, C(m)=C(n)=2$ for $m,n\neq 0$

Also involving cosine functions only!

- Formula (**) can be expressed as matrix multiplication:

$$f=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}_{ec}(m,n)\vec{T}_m\vec{T'}_n^T$$

elementary images under EDCT!

where: $\vec{T}_m=\begin{pmatrix}T_m(0)\\T_m(1)\\\vdots\\T_m(M-1)\end{pmatrix}, \vec{T'}_n=\begin{pmatrix}T'_n(0)\\T'_n(1)\\\vdots\\T'_n(N-1)\end{pmatrix}$ with $T_m(k)=C(m)\cos\frac{\pi m(2k+1)}{2M}$

and $T'_n(k)=C(n)\cos\frac{\pi n(2k+1)}{2N}$.

This is what JPEG does!!

Something similar can be developed:

<u>Definition: (Odd symmetric discrete cosine transform [ODCT])</u>

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M-1$ and $0 \leq l \leq N-1$. The **odd symmetric discrete cosine transform (ODCT)** of f is given by:

$$\hat{f}_{oc}(m,n) = \frac{1}{(2M-1)(2N-1)} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} C(k)C(l)f(k,l) \cos\frac{2\pi mk}{2M-1} \cos\frac{2\pi nl}{2N-1}$$

where $C(0) = 1$ and $C(k) = C(l) = 2$ for $k, l \neq 0$, $0 \leq m \leq M-1, 0 \leq n \leq N-1$.

The **inverse ODCT** is given by:

$$f(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{oc}(m,n) \cos\frac{2\pi mk}{2M-1} \cos\frac{2\pi nl}{2N-1}$$

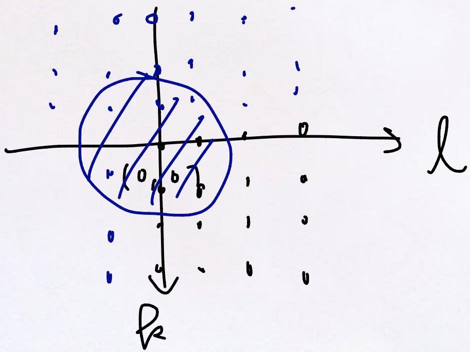
where $C(0) = 1, C(m) = C(n) = 2$ if $m, n \neq 0$

Understanding convolution:

Recall: Discrete convolution:

$$\underset{g * I(n,m)}{\underset{\shortparallel}{v(n,m)}} = \underbrace{\sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m') \, I(n',m')}_{\text{Linear combination of pixel values of } I}$$

In particular, if $g(k,l)$ is only non-zero around $(0,0)$, then, $g * I(n,m)$ is a linear combination of pixel value of I around (n,m)!!



Example: Suppose g looks like the following:

$$g = \left(\begin{array}{ccccc} & 1 & 2 & 3 & \\ 0 & 4 & 5 & 6 & 0 \\ & 7 & 8 & 9 & \end{array} \right) \begin{array}{l} \leftarrow k=-1 \\ \leftarrow k=0 \\ \leftarrow k=1 \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ l=-1 & l=0 & l=1 \end{array}$$

$$I * g(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, I(n',m')$$

Linear combination of neighborhood pixel values

$$\begin{aligned} &= 1 \cdot I(n+1, m+1) + 2 \cdot I(n+1, m) + 3 \cdot I(n+1, m-1) \\ &+ 4 \cdot I(n, m+1) + 5 \cdot I(n, m) + 6 \cdot I(n, m-1) \\ &+ 7 \cdot I(n-1, m+1) + 8 \cdot I(n-1, m) + 9 \cdot I(n-1, m-1) \end{aligned}$$

Note:

(Spatial domain) $I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\downarrow \text{DFT}$$

(Frequency domain) $MN\,\underbrace{\hat{I} \odot}_{\text{pixel-wise multiplication}}\,\hat{g}$ (Modifying the Fourier coefficients by multiplication)