Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

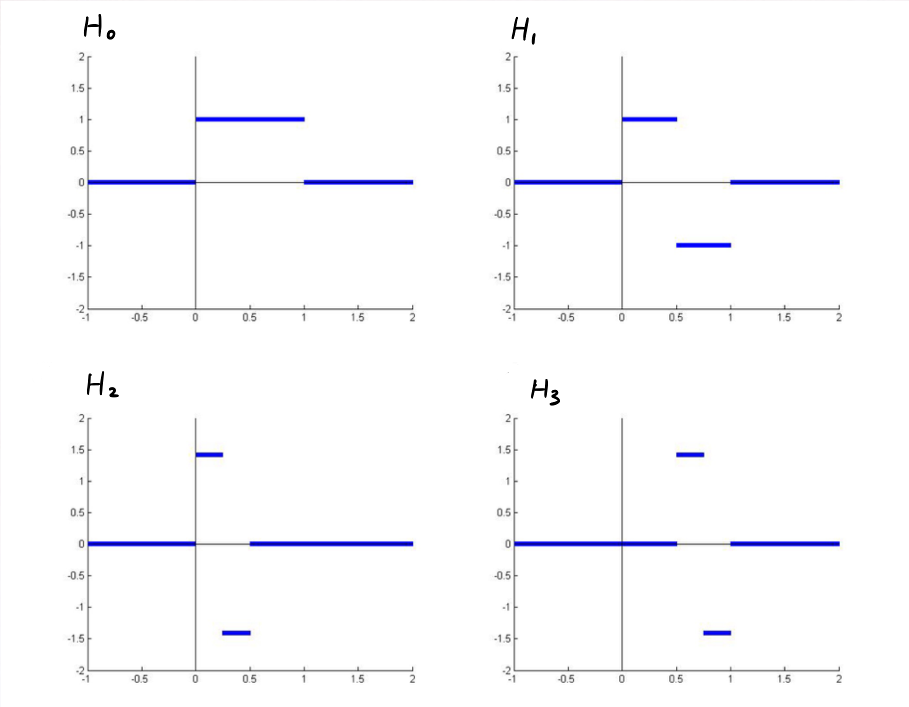
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

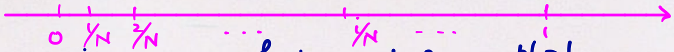
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \dots, N-1$.

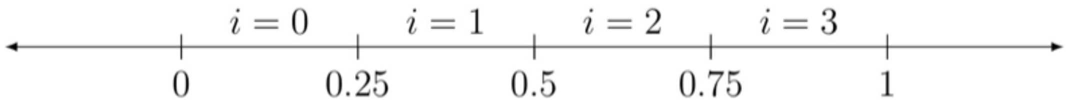
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \big(H(k,i)\big)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

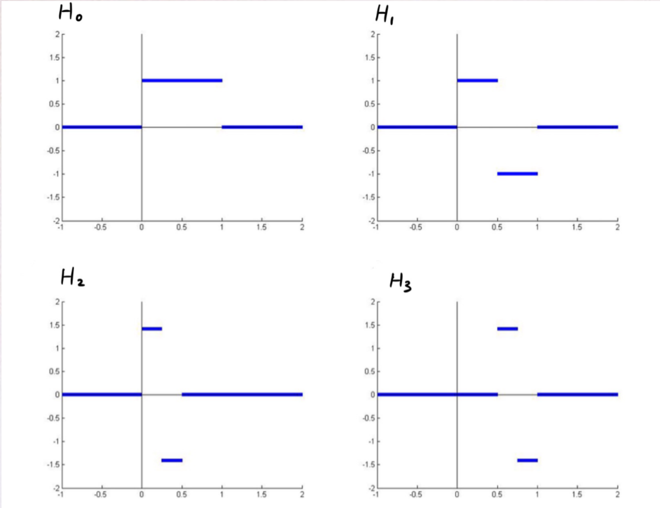
$$g = \tilde{H} f \tilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}}H = \frac{1}{2}H$$

Easy to check that $\tilde{H}^T\tilde{H} = I$.

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

More zeros

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T f \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

g ← transformed image

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_1^T & - \\ - & \vec{h}_2^T & - \\ & \vdots & \\ - & \vec{h}_N^T & - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{I_{ij}^H}$

I_{ij}^T = elementary images under Haar Transform.

Definition: (Walsh function) The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \dots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

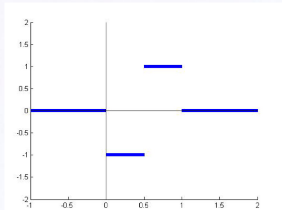
Example: Compute $W_1(t)$.

Put $j = 0$, $q = 1$. Then:

$$W_1(t) = (-1)^{\lfloor 0 \rfloor + 1} \{ W_0(2t) + (-1)^1 W_0(2t-1) \} = (-1) \{ W_0(2t) + (-1)^1 W_0(2t-1) \}$$

For $0 \le t < \frac{1}{2}$, $W_0(2t) = 1$, $W_0(2t-1) = 0 \Rightarrow W_1(t) = -1$.

For $\frac{1}{2} \le t < 1$, $W_0(2t) = 0$, $W_0(2t-1) = 1 \Rightarrow W_1(t) = 1$.



Definition: (Discrete Walsh transform)

The Walsh Transform of a $N \times N$ image is defined as follows.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \dots, N-1$.

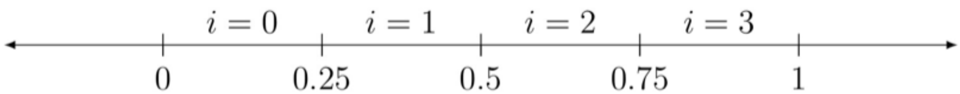
The Walsh transform matrix is: $\tilde{W} \equiv \frac{1}{\sqrt{N}} W$ where $W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

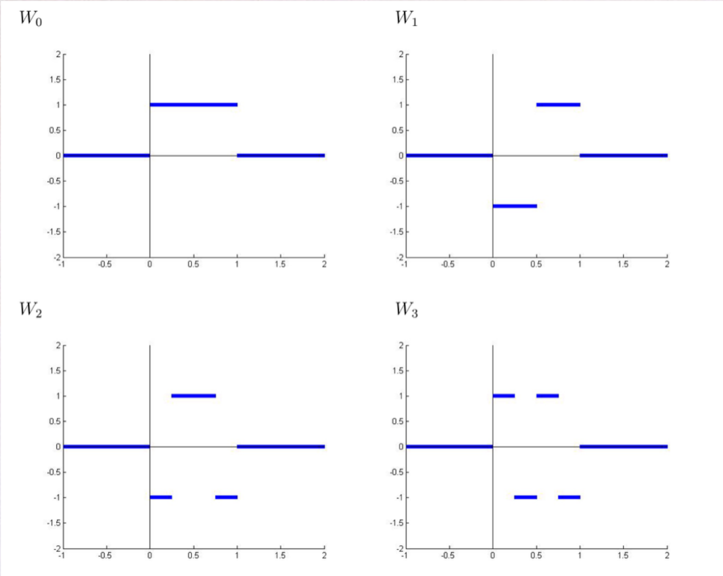
$$g = \tilde{W} f \tilde{W}^T$$

Example Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:



We can check that:



So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}} W = \frac{1}{2} W$$

$(\tilde{W}^T \tilde{W} = I)$

Example 2.7: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{W} f \tilde{W}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

More zeros in the coefficient matrix!

Remark: 1. Walsh transform is to transform an image to a "transformed image" with much more zeros.

Elementary images under Walsh transform:

Under Walsh Transform, $f = \tilde{W}^T g \tilde{W}$.

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{W}_i \vec{W}_j^T$ where $\tilde{W} = \begin{pmatrix} - \vec{W}_1^T - \\ - \vec{W}_2^T - \\ \vdots \\ - \vec{W}_N^T - \end{pmatrix}$

" I_{ij}^W

I_{ij}^W = elementary images under Walsh transform.