Lecture 2:

Theorem: Suppose PSF is separable. Then:

1. The operator $\mathcal{O}$ consists of two matrix multiplication.
2. If PSF is also shift-invariant, then the operator $\mathcal{O}$ consists of two 1-D convolutions.

Proof: 1.

$$g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta)$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h_c(x,\alpha)\, h_r(y,\beta)$$

$$= \underbrace{\sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\left(\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)\right)}_{\text{matrix multiplication}}}_{\text{matrix multiplication}}$$

$$2.\quad g(\alpha,\beta) = \underbrace{\sum_{x=1}^{N} h_c(\alpha - x) \underbrace{\sum_{y=1}^{N} h_r(\beta - y) f(x,y)}_{\text{1-D convolution}}}_{\text{1-D convolution}}$$

Shift-invariant:

$$h(x,\alpha,y,\beta) = h(\alpha - x, \beta - y)$$

Representation of $\mathcal{O}$ by a matrix H:

We can write:

$$g(\alpha,\beta) = f(1,1)\,h(1,\alpha,1,\beta) + f(2,1)\,h(2,\alpha,1,\beta) + \dots + f(N,1)\,h(N,\alpha,1,\beta)$$
$$+ f(1,2)\,h(1,\alpha,2,\beta) + \dots + f(N,2)\,h(N,\alpha,2,\beta)$$
$$\dots$$
$$+ f(1,N)\,h(1,\alpha,N,\beta) + \dots + f(N,N)\,h(N,\alpha,N,\beta)$$

$$= \vec{h}_{\alpha,\beta}^{\,T} \cdot \vec{f}^{\,T} = \begin{pmatrix} f(1,1) \\ \vdots \\ f(N,1) \\ f(1,2) \\ \vdots \\ f(N,2) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} = \vec{f}$$

$$\vec{h}_{\alpha,\beta} = \begin{pmatrix} h(1,\alpha,1,\beta) \\ \vdots \\ h(N,\alpha,1,\beta) \\ h(1,\alpha,2,\beta) \\ \vdots \\ h(N,\alpha,2,\beta) \\ \vdots \\ h(1,\alpha,N,\beta) \\ \vdots \\ h(N,\alpha,N,\beta) \end{pmatrix}$$

In matrix form, let

$$\vec{g} = \begin{pmatrix} g(1,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix}.$$

Then: $\vec{g} = H\vec{f}$

$N^2 \times N^2$ matrix

By careful examination, we see that:

$$H = \begin{pmatrix}
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = 1 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = 2 \end{pmatrix} \end{pmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = N \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = N \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = N \end{pmatrix} \end{pmatrix}
\end{pmatrix}$$

Meaning of

$$\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = i \\ \beta = j \end{pmatrix} \end{pmatrix} : \begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix}$$

$$\in M_{N \times N}$$

<u>Definition</u>: H is called the <u>transformation matrix</u> of $\mathcal{O}$.

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$3 \times 3 \text{ image} = \begin{array}{c c c} & \begin{array}{ccc} f_{31} & f_{32} & f_{33} \end{array} & \\ \begin{array}{c} f_{13} \\ f_{23} \\ f_{33} \end{array} & \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} & \begin{array}{c} f_{11} \\ f_{21} \\ f_{31} \end{array} \\ & \begin{array}{ccc} f_{11} & f_{12} & f_{13} \end{array} & \end{array}$$

$$g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \; ; \quad g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}$$

etc ...

By careful examination, we see that

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

What is $h(2,3,2,1)$?
What is $h(1,2,3,3)$?

$h(2,3,2,1) = 0$

$h(1,2,3,3) = 1/4$.

Recall:

$$H = \begin{pmatrix} \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=1 \end{pmatrix}}\right) \\ \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=2 \end{pmatrix}}\right) \\ \vdots & \vdots & & \vdots \\ \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=1 \\ \beta=N \end{pmatrix}}\right) & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=2 \\ \beta=N \end{pmatrix}}\right) & \cdots & \left(\alpha\downarrow \overset{x\rightarrow}{\begin{pmatrix} y=N \\ \beta=N \end{pmatrix}}\right) \end{pmatrix}$$

Example 1.2 Consider an image transformation on a 2×2 image. Suppose the matrix representation of the image transformation is given by:

$$H = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 \\ 3 & 0 & 4 & 0 \\ 6 & 3 & 8 & 4 \end{pmatrix}.$$

Prove that the image transformation is separable. Find g_1 and g_2 such that:

$$h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta).$$

$g_1(1,1)\, g_2(1,1)$ $=$

$$\begin{pmatrix} h(1,1,1,1) & h(2,1,1,1) \\ h(1,2,1,1) & h(2,2,1,1) \end{pmatrix}$$

Solution: H for a 2×2 image:

$$\begin{pmatrix} \overset{x \rightarrow}{\alpha\downarrow\begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}} & \overset{\rightarrow}{\alpha\downarrow\begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}} \\ \overset{x \rightarrow}{\downarrow\begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}} & \overset{x \rightarrow}{\alpha\downarrow\begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}} \end{pmatrix} \in M_{4\times 4}$$

H is separable $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g_1(x,\alpha)\, g_2(y,\beta)$.

Easy to check:

if H is separable: $H = \begin{pmatrix} g_2(1,1)\, G_1 & g_2(2,1)\, G_1 \\ g_2(1,2)\, G_1 & g_2(2,2)\, G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_1(2,2) \end{pmatrix}$

In our case, $H = \begin{pmatrix} 2G_1 & 1G_1 \\ 3G_1 & 4G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\therefore$

$$g_1(1,1) = 1 \qquad g_2(1,1) = 2$$
$$g_1(2,1) = 0 \qquad g_2(2,1) = 1$$
$$g_1(1,2) = 2 \qquad g_2(1,2) = 3$$
$$g_1(2,2) = 1 \; // \qquad g_2(2,2) = 4 \; //$$

Example 1.3 Suppose $H \in M_{4\times 4}$ is applied to a 2×2 image. Let

$$H = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 4 \\ 2 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \end{pmatrix}.$$

Is H shift-invariant?

Solution: Shift-invariant $\iff h(x,\alpha,y,\beta) = h(\alpha - x, \beta - y)$

Easy to check that: $h(\underbrace{1,2}_{1}\ \underbrace{1,1}_{0}) = 2$

$\neq$

$h(\underbrace{1,2}_{1},\ \underbrace{2,2}_{0}) = 1$

$\therefore$ Not shift-invariant!

$$\alpha \downarrow \overset{\xrightarrow{x}}{\begin{pmatrix} y = i \\ \beta = j \end{pmatrix}}$$

1st row - 2nd col block

1st row 1st col

Properties of shift-invariant/separable image transformation

Definition: (Circulant matrix)

A circulant matrix $V := \mathrm{circ}(\vec{v})$ associated to a vector $\vec{v} = (v_0, v_1, \ldots, v_{n-1})^T \in \mathbb{C}^n$ is a $n \times n$ matrix whose columns are given by iterations of shift operator T acting on $\vec{v}$. Here, $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_0 \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix}.$$

$\therefore$ k^{th} column is given by $T^{k-1}(\vec{v})$ $(k = 1, 2, \ldots, n)$

$$\therefore \quad V = \begin{pmatrix} v_0 & v_{n-1} & v_{n-2} & \cdots & v_1 \\ v_1 & v_0 & v_{n-1} & \cdots & v_2 \\ \vdots & \vdots & v_0 & & \vdots \\ & & \vdots & & v_{n-1} \\ v_{n-1} & v_{n-2} & v_{n-3} & \cdots & v_0 \end{pmatrix}$$

Definition:(Block circulant)

V is block-circulant $\Leftrightarrow$ $V = \begin{pmatrix} \overset{N\times N}{H_0} & H_{n-1} & \cdots & H_1 \\ \overset{N\times N}{H_1} & H_0 & & H_2 \\ \vdots & \vdots & \cdots & \vdots \\ & & & H_{n-1} \\ \overset{N\times N}{H_{n-1}} & H_{n-2} & \cdots & H_0 \end{pmatrix}$ $\ni$

each H_i is a circulant matrix.

Theorem: If H = transf. matrix of shift-invariant operator,

then $H = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ A_{21} & \cdots & A_{2N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$ where each A_{ij} is a circulant matrix.

Theorem: H is block circulant!

(Exercise)