Lecture 17:

Image denoising using energy minimization

Let g be a noisy image corrupted by additive noise n.

Then: $g(x,y) = \underbrace{f(x,y)}_{\text{Clean image}} + \underbrace{n(x,y)}_{\text{noise}}$

Recall: Laplacian masking : $g = f - \Delta f$ (Obtain a (non-smooth) sharp image from a smooth image)

Conversely, to get a smooth image f from a non-smooth image g, we can solve the PDE for f : $-\Delta f + f = \underbrace{g}_{\text{known}}$

unknown

We will show that solving the above equation is equivalent to minimizing something :

$$E(f) = \iint \left(f(x,y) - g(x,y)\right)^2 dx\,dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy$$

In the discrete case, the PDE can be approximated (discretized) to get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)]$$

for all (x,y) (Linear System)

Solved by

Direct method (Big linear system)

Iterative method

Simple iterative scheme: Let g be an $N \times N$ image.

Step 1: Let $f^0(x,y) = g(x,y)$ (Initial guess of the solution)

Step 2: For $n \geq 0$ and for all $(x,y), x = 1, \cdots, M, y = 1, \cdots, N$,

$$f^{n+1}(x,y) = g(x,y) + [f^n(x+1,y) + f^n(x-1,y) + f^n(x,y+1) + f^n(x,y-1) - 4f^n(x,y)]$$

Impose boundary conditions by reflection:

$$\begin{aligned}
&f^{n+1}(0,y) = f^{n+1}(2,y); \quad f^{n+1}(M+1,y) = f^{n+1}(M-1,y) \quad \text{for } y = 1, \cdots, N \\
&f^{n+1}(x,0) = f^{n+1}(x,2); \quad f^{n+1}(x,N+1) = f^{n+1}(x,N-1) \quad \text{for } x = 1, \cdots, M \\
&f^{n+1}(0,0) = f^{n+1}(2,2); \quad f^{n+1}(0,N+1) = f^{n+1}(2,N-1) \\
&f^{n+1}(M+1,0) = f^{n+1}(M-1,2); \quad f^{n+1}(M+1,N+1) = f^{n+1}(M-1,N-1)
\end{aligned}$$

(similar for f^n)

Step 3: Continue the process until $||f^{n+1} - f^n|| \leq$ tolerance. (Convergence depends on the spectral radius of a matrix)

Consider : $E_{discrete}(f) = \sum_{x=1}^{N} \sum_{y=1}^{N} (f(x,y) - g(x,y))^2 + \sum_{x=1}^{N} \sum_{y=1}^{N} \left[(f(x+1,y) - f(x,y))^2 + (f(x,y+1) - f(x,y))^2 \right]$

Suppose f is a minimizer of $E_{discrete}$. Then, for each (x,y),

$E_{discrete}$ depends on $f(x,y)$ for each (x,y)

$$\frac{\partial E_{discrete}}{\partial f(x,y)} = 0.$$

$$\text{"} \; 2(f(x,y) - g(x,y)) + 2(f(x+1,y) - f(x,y))(-1) + 2(f(x,y+1) - f(x,y))(-1)$$
$$+ 2(f(x,y) - f(x-1,y)) + 2(f(x,y) - f(x,y-1))$$

By simplification, we get :

$$f(x,y) = g(x,y) + \left[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \right]$$

The continuous version of $E_{discrete}$ can be written as :

$$E(f) = \iint (f(x,y) - g(x,y))^2 + \iint \Big[\underbrace{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}_{|\nabla f|^2} \Big] dx\,dy$$

Remark:

- Solving $f = g + \Delta f$ is equivalent to energy minimization
- The first term in Ediscrete is called the fidelity term.
  Aim to find f that is close to g.
- The second term is called the regularization term. Aim to enhance smoothness.
- $-\nabla f + f = g$ can also be solved in the frequency domain:

$$DFT(f) = DFT(g + \underbrace{\Delta f}_{p * f})$$

$$\therefore\ DFT(f)(u,v) = DFT(g)(u,v) + c\,DFT(p)(u,v)\,DFT(f)(u,v)$$

$$\Longleftrightarrow\ DFT(f)(u,v) = \left[\frac{1}{1 - c\,DFT(p)(u,v)}\right] DFT(g)(u,v)$$

$\Big\downarrow$ inverse DFT

$f(x,y)$!!