Lecture 14: Recall:

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$, $S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y)$ = inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We'll show: Wiener filter minimizes the mean square error:

$$\varepsilon^2(\underset{\text{original}}{f}, \underset{\text{Restored}}{\hat{f}}) = \iint |f(x,y) - \hat{f}(x,y)|^2$$

(We assume the continuous case to avoid complicated indices)

Let $\underset{\text{Observed}}{g} = \overset{\text{degradation}}{h} * \underset{\text{original}}{f} + \underset{\text{noise}}{n}$

Assume that f and n are spatially uncorrelated:

$$\iint f(x,y)\, n(x+r, y+s) \quad \text{for all } r, s.$$

Define: $\hat{f}(x,y) = w(x,y) * g(x,y)$ for some $w(x,y)$

(FT of $\hat{f}$ is like: $W(u,v)\, G(u,v)$)

Goal: Find $W(u,v)$ such that $\varepsilon^2(f, \hat{f})$ is minimized.

Recall: $\hat{f}$ is obtained as follows:

Step 1: Let $\hat{F}(u,v) = \underbrace{W(u,v)}_{\text{Filter}}\, G(u,v)$

Step 2: Compute iFT of $\hat{F}$ to get $\hat{f}$

$\therefore\ \hat{f} = w * g$ for some w.

(Sketch of proof)

We need to use: Parseval Theorem:

$$\varepsilon^2(f,\hat{f}) := \iint |f(x,y) - \hat{f}(x,y)|^2 = C \iint |F(u,v) - \hat{F}(u,v)|^2 \quad \text{for some constant } C$$

where $F(u,v)$ = DFT of f and $\hat{F}(u,v)$ = DFT of $\hat{f}$

Let $G(u,v)$ = DFT of $\overset{\text{observed image}}{g}$ and $N(u,v)$ = DFT of n

Then: $\hat{F}(u,v) = W(u,v)\,G(u,v)$ (as $\hat{f}(x,y) = iFT(W(u,v)\,G(u,v))$)

So, $\hat{F}(u,v) = W(u,v)\,G(u,v) = W(u,v)\big(H(u,v)\,F(u,v) + N(u,v)\big)$

In other words, $F - \hat{F} = (1 - WH)F - WN$

and $\varepsilon^2(f,\hat{f}) = C \iint |(1-WH)F - WN|^2$

Since f and n are spatially uncorrelated, we can show that:

$$\mathcal{E}^2(f, \hat{f}) = \iint |(1-WH)F|^2 + |WN|^2$$

$$\left(\iint (1-WH)F\,\overline{W}\,\overline{N} \qquad = 0 \right)$$

Since we look for $w(x,y)$ (hence $W(u,v)$) such that $\mathcal{E}^2$ is minimized, we can regard $\mathcal{E}^2$ is dependent on W.

To minimize $\mathcal{E}^2(W)$, we consider:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}^2(W + tV) = 0 \text{ for all } V.$$

We get: $\iint -(1-\overline{W}\,\overline{H})H|F|^2 V - (1-WH)\overline{H}|F|^2\overline{V} + \overline{W}|N|^2 V + W|N|^2\overline{V} = 0$ for all V.

Put $V = -(1-WH)\overline{H}|F|^2 + W|N|^2$. Then: we have: $\iint \left|-(1-WH)\overline{H}|F|^2 + W|N|^2\right|^2 \qquad = 0.$

$$\therefore \quad -(1-WH)\bar{H}|F|^2 + W|N|^2 = 0$$

$$\Downarrow$$

$$W = \frac{\bar{H}}{|H|^2 + |N|^2/|F|^2}.$$

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underset{\text{degradation}}{h} * f + \underset{\text{noise}}{n}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$ $\qquad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

$\mathcal{S}(g)$: Stacked image of g

D: transformation matrix of $h * f$ (or $\vec{f}$)

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \text{ subject to the constraint: }$$

$$\|\vec{g} - H\vec{f}\|^2 = \varepsilon$$

- $\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \longleftarrow$ Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon \longleftarrow$ Deblur

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \xrightarrow{\text{Put } h=1} \nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

More generally, $\nabla^2 f = p * f \longleftarrow$ discrete convolution

where

$$p = \begin{array}{cc} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix} & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & x=0 \\ y=0 & \end{array}$$

<u>Remark:</u> $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ means we Allow some fixed level of noise.

$\qquad\qquad = \|\vec{n}\|^2$

Assume $S(p * f) = L\vec{f}$ transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T (L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$\left(D^T D + \gamma L^T L\right)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\,G(u,v)$$

($H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$)

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{H(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Sketch of proof:

Recall: our problem is to minimize:

$$\vec{f}^T L^T L \vec{f} \text{ subject to } \|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

$$\| \vec{g} - D\vec{f} \|^2 = (\vec{g} - D\vec{f})^T (g - D\vec{f})$$

From calculus, the minimizer must satisfy:

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}} \left(\vec{f}^T L^T L \vec{f} + \lambda (\vec{g} - D\vec{f})^T (\vec{g} - D\vec{f}) \right) = 0 \quad \text{for}$$

where $\vec{f} = (f_1, f_2, \dots, f_i, \dots, f_{N^2})^T$ and λ is the Lagrange's multiplier.

Here, $\frac{\partial K}{\partial \vec{f}} = \left(\frac{\partial K}{\partial f_1}, \frac{\partial K}{\partial f_2}, \dots, \frac{\partial K}{\partial f_{N^2}} \right)^T$

Easy to check:

- $\frac{\partial (\vec{f}^T \vec{a})}{\partial \vec{f}} = \vec{a}$
- $\frac{\partial (\vec{b}^T \vec{f})}{\partial \vec{f}} = \vec{b}$
- $\frac{\partial (\vec{f}^T A \vec{f})}{\partial \vec{f}} = (A + A^T) \vec{f}$

$$\vec{f}^T \vec{a} = (f_1, f_2, \ldots, f_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 f_1 + a_2 f_2 + \ldots + a_n f_n$$

$$\frac{\partial \vec{f}^T \vec{a}}{\partial f_j} = a_j$$

$$\therefore \frac{\partial \vec{f}^T \vec{a}}{\partial \vec{f}} \overset{\text{def}}{=} \left(\frac{\partial \vec{f}^T \vec{a}}{\partial f_1}, \ldots, \frac{\partial \vec{f}^T \vec{a}}{\partial f_n} \right)^T = (a_1, a_2, \ldots, a_n)^T$$

etc. . ..

$\therefore \; D = 0 \Rightarrow (2L^T L)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^T D\vec{f}) = 0$

$\Rightarrow (D^T D + \gamma L^T L)\vec{f} = D^T\vec{g}$ where $\gamma = \frac{1}{\lambda}$ and λ is the Lagrange's multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f}) = \varepsilon.$$

Now, we'll consider the frequency domain.

Note that D and L are transformation matrix of convolution.

$\therefore$ D and L are block-circulant.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Using the fact that both D and L are block-circulant, we can check that:

Fact 1:

$$D = W\Lambda_D W^{-1},\ D^T = W\Lambda_D^* W^{-1},\ L = W\Lambda_L W^{-1},\ L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,

$$\Lambda_D(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

DFT(h)

where $\ddot{H} = DFT(h)$.

and

$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & & \cdots & & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$\Lambda_D = \begin{pmatrix} N^2 H(0,0) & & & & & & & \\ & N^2 H(1,0) & & & & & & \\ & & \ddots & & & & & \\ & & & N^2 H(N-1,0) & & & & \\ & & & & N^2 H(1,1) & & & \\ & & & & & \ddots & & \\ & & & & & & N^2 H(N-1,1) & \\ & & & & & & & \ddots \end{pmatrix}$$

1st col of H becomes first N diagonal entries

2nd col of H becomes 2nd N diagonal entries