MATH3360: Mathematical Imaging

Assignment 3

Due: 14 November 2021

1. (a) Consider a $2N \times 2N$ image $I = (I(m,n))_{-N \leq m,n \leq N-1}$. The Butterworth high-pass filter H of squared radius D_0 and order n is applied on $DFT(I) = (\hat{I}(u,v))_{-N \leq u,v \leq N-1}$ to give $G(u,v)$. Suppose $\hat{I}(2,0) \neq 0$ and $\hat{I}(-1,3) \neq 0$, and
$$G(2,0) = \frac{1}{65}\hat{I}(2,0) \text{ and } G(-1,3) = \frac{25}{281}\hat{I}(-1,3).$$
Find D_0 and n.

   (b) Consider a $(2M+1) \times (2N+1)$ image $I = (I(m,n))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$, where $M, N > 200$. The Gaussian high-pass filter with standard deviation σ is applied to $DFT(I) = (\hat{I}(u,v))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$. Suppose $H(2,2) = \dfrac{1}{MN}$. Find σ^2.

2. Let $h = (h(k,l))_{0 \leq k,l \leq N-1}$ and $H \in M_{N^2 \times N^2}$ such that for any periodically extended $f \in M_{N \times N}$, $\mathcal{S}(h * f) = H\mathcal{S}(f)$, where $\mathcal{S}$ is the stacking operator. Show that $H(x,y) = h(mod_N(x) - mod_N(y), \lfloor \frac{x}{N} \rfloor - \lfloor \frac{y}{N} \rfloor)$.

3. (*Optional*) Let $W_N(n,k) = \dfrac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$ for $0 \leq n,k \leq N-1$ and $W = W_N \otimes W_N$.

   (a) Prove that $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

   (b) Show that $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N \times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

4. The discrete Gaussian blur operator $\mathcal{G}$ on a periodically extended $N \times N$ image ($N \geq 3$) can be written as:
$$\begin{aligned}\mathcal{G}f(x,y) = &\frac{1}{16}f(x+1,y+1) + \frac{1}{8}f(x+1,y) + \frac{1}{16}f(x+1,y-1)\\ &+ \frac{1}{8}f(x,y+1) + \frac{1}{4}f(x,y) + \frac{1}{8}f(x,y-1)\\ &+ \frac{1}{16}f(x-1,y+1) + \frac{1}{8}f(x-1,y) + \frac{1}{16}f(x-1,y-1).\end{aligned}$$
Prove that $DFT(\mathcal{G}f)(u,v) = G(u,v)F(u,v)$ for some $G \in M_{N \times N}(\mathbb{C})$, where $F = DFT(f)$. Find $G(u,v)$ as a trigonometric polynomial in $\frac{\pi u}{N}$ and $\frac{\pi v}{N}$, i.e. as a polynomial in $\sin\frac{\pi u}{N}$, $\cos\frac{\pi u}{N}$, $\sin\frac{\pi v}{N}$ and $\cos\frac{\pi v}{N}$.

5. Suppose $g \in M_{N \times N}(\mathbb{R})$ is a blurred image capturing a static scene. Assume that g is given by:
$$g(i,j) = \frac{1}{\lambda}\sum_{k=0}^{\lambda-1} f(i-k, j-k) \text{ for } 0 \leq i,j \leq N-1,$$
where $\lambda \in \mathbb{N} \cap [1,N]$ and f is the underlying image (periodically extended). Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ for all $0 \leq u,v \leq N-1$, where $H(u,v)$ is the degradation function in the frequency domain given by:
$$H(u,v) = \begin{cases} \frac{1}{\lambda}\frac{\sin\frac{\lambda\pi(u+v)}{N}}{\sin\frac{\pi(u+v)}{N}} e^{-\pi j \frac{(\lambda-1)(u+v)}{N}} & \text{if } u+v \notin \{0,N\},\\ 1 & \text{if } u+v \in \{0,N\}.\end{cases}$$

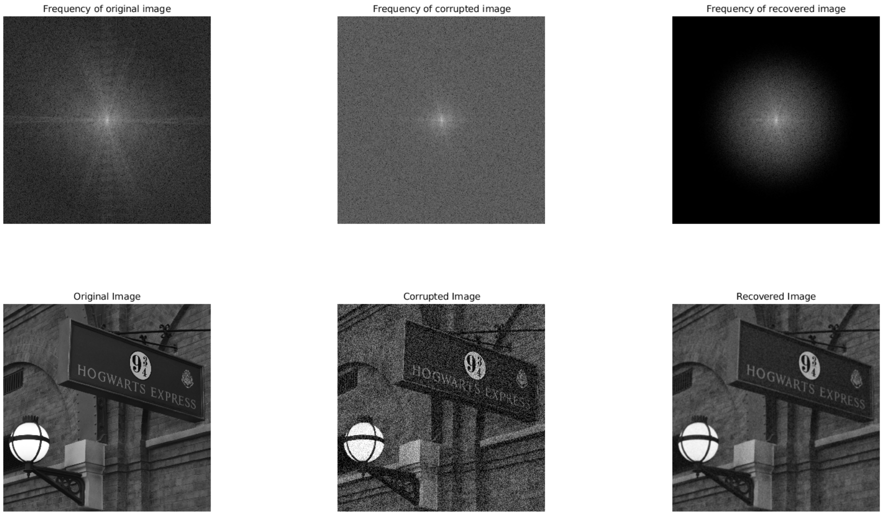

Figure 1: Experimental Results

6. **Coding assignment**: Please read the MATLAB file in the attached zip file carefully. There are missing lines in the file. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

   Coding instruction: In HW2, we implement DFT hand by hand. In this HW, we use the built-in function $fft2(ifft2)$ and $fftshift(ifftshift)$ to simplify our code. The only difference between $fft2$ and our definition of DFT is that our definition is $\frac{1}{N^2}$ of $fft2$. Similarly, Our definition of iDFT is N^2 of $ifft2$.

   The built-in function $fftshift$ and $ifftshift$ have the same effect as $circshift(x, [h/2, w/2])$, which shifts frequency components at the corner to the center. So we use the two functions to replace $circshift$ to simplify the code.

   In order to compute the distance between each element and the center, we make use of the built-in function $meshgrid$. Here is an example of the function $[X, Y] = meshgrid([-2 : 2], [-1 : 1])$:

   $$X = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \qquad Y = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

   You are required to define the Gaussian Low Pass Filter using the definition

   $$H(u, v) = e^{-\frac{u^2+v^2}{2\sigma^2}}$$

   where $-\frac{N}{2} \leq u \leq \frac{N}{2} - 1$ and $-\frac{N}{2} \leq v \leq \frac{N}{2} - 1$.

   The results are as follows