# MATH3360: Mathematical Imaging

## Assignment 3 Solutions

1. (a) The given information implies

$$H(2,0) = \frac{1}{65} \text{ and } H(-1,3) = \frac{25}{281}.$$

Hence

$$\begin{cases} \dfrac{4^n}{D_0^{2n} + 4^n} &= \dfrac{1}{65} \\ \dfrac{10^n}{D_0^{2n} + 10^n} &= \dfrac{25}{281}. \end{cases} \text{ and thus } D_0^{2n} = 64 \cdot 4^n = \frac{256}{25} \cdot 10^n.$$

Then $(\frac{10}{4})^n = 64 \cdot \frac{25}{256}$ which gives $n = 2$, and thus $D_0^4 = 1024$ which gives $D_0 = 4\sqrt{2}$.

(b) Carefully note that we do not assume central spectrum in this question. So, the entries near $(M, N)$ are high frequency components. Performing a translation, we have

$$exp\big(-\frac{(M-2)^2 + (N-2)^2}{2\sigma^2}\big) = \frac{1}{MN}$$

Hence, $\sigma^2 = \dfrac{(M-2)^2 + (N-2)^2}{2 \log MN}$

2. For any $x \in \mathbb{N} \cap [0, N^2 - 1]$,

$$\begin{aligned} \mathcal{S}(g)(x) &= g(mod_N(x), \lfloor \frac{x}{N} \rfloor) \\ &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) h(mod_N(x) - k, \lfloor \frac{x}{N} \rfloor - l) \\ &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h(mod_N(x) - k, \lfloor \frac{x}{N} \rfloor - l) \mathcal{S}(f)(k + lN). \end{aligned}$$

Hence for any $x \in \mathbb{N} \cap [0, N^2 - 1]$ and $k, l \in \mathbb{N} \cap [0, N - 1]$,

$$H(x, k + lN) = h(mod_N(x) - k, \lfloor \frac{x}{N} \rfloor - l)$$

and thus for any $x, y \in \mathbb{N} \cap [0, N^2 - 1]$,

$$H(x, y) = h(mod_N(x) - mod_N(y), \lfloor \frac{x}{N} \rfloor - \lfloor \frac{y}{N} \rfloor).$$

3. (a) For any $x, \alpha \in \mathbb{Z} \cap [0, N-1]$,

$$
\begin{aligned}
(W_N\overline{W_N})(\alpha, x) &= \sum_{k=0}^{N-1} W_N(\alpha, k)\overline{W_N(k, x)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi j\frac{\alpha k}{N}} e^{-2\pi j\frac{kx}{N}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi j\frac{k(\alpha-x)}{N}} \\
&= \frac{1}{N}\cdot N\mathbf{1}_{N\mathbb{Z}}(\alpha - x) = \delta(\alpha - x).
\end{aligned}
$$

Hence $W_N\overline{W_N} = I_N$, the $N\times N$ identity matrix, and $W_N^{-1} = \overline{W_N}$. Then denoting the $(\beta, y)$-th block of a $N\times N$ matrix $A$ of $N\times N$ blocks by $A_{\beta,y}$, and denoting $Y = \overline{W_N}\otimes\overline{W_N}$, we have:

$$
\begin{aligned}
(WY)_{\beta,y} &= \sum_{k=0}^{N-1} W_{\beta,k}Y_{k,y} = \sum_{k=0}^{N-1} W_N(\beta, k)W_N\cdot\overline{W_N(k, y)}\,\overline{W_N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi j\frac{k(\beta-y)}{N}} I_N = \frac{1}{N}\cdot N\mathbf{1}_{N\mathbb{Z}}(\beta - y)I_N = \delta(\beta - y)I_N.
\end{aligned}
$$

Hence $WY = I_{N^2\times N^2}$, the $N^2\times N^2$ identity matrix, and thus $\overline{W_N}\otimes\overline{W_N} = W^{-1}$.

(b) For any $x \in \mathbb{N}\cap[0, N^2-1]$,

$$
\begin{aligned}
W^{-1}\mathcal{S}(f)(x) &= \sum_{k=0}^{N^2-1}(\overline{W_N}\otimes\overline{W_N})(x, k)\mathcal{S}(f)(k) \\
&= \frac{1}{N}\sum_{k=0}^{N^2-1}\overline{W_N(\lfloor\frac{x}{N}\rfloor, \lfloor\frac{k}{N}\rfloor)}\,\overline{W_N(mod_N(x), mod_N(k))} f(mod_N(k), \lfloor\frac{k}{N}\rfloor) \\
&= \frac{1}{N}\sum_{k=0}^{N^2-1} e^{-2\pi j\frac{\lfloor\frac{x}{N}\rfloor\lfloor\frac{k}{N}\rfloor + mod_N(x)mod_N(k)}{N}} f(mod_N(k), \lfloor\frac{k}{N}\rfloor) \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-2\pi j\frac{n\lfloor\frac{x}{N}\rfloor + m\, mod_N(x)}{N}} f(m, n) \\
&= N\hat{f}(mod_N(x), \lfloor\frac{x}{N}\rfloor) = N\mathcal{S}(\hat{f})(mod_N(x) + N\lfloor\frac{x}{N}\rfloor) = N\mathcal{S}(\hat{f})(x).
\end{aligned}
$$

Hence $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$.

4. Note that $\mathcal{G}(f) = h * f$, where

$$h(x,y) = \begin{cases} \frac{1}{4} & \text{if } x^2+y^2=0 \\ \frac{1}{8} & \text{if } x^2+y^2=1 \\ \frac{1}{16} & \text{if } x^2+y^2=2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $DFT(\mathcal{G}(f)) = N^2 DFT(h) \odot DFT(f)$, and thus

$$\begin{aligned} G(u,v) &= N^2 DFT(h)(u,v) \\ &= \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} h(x,y) e^{-2\pi j \frac{ux+vy}{N}} \\ &= \frac{1}{4} + \frac{1}{8}(e^{2\pi j \frac{u}{N}} + e^{-2\pi j \frac{u}{N}} + e^{2\pi j \frac{v}{N}} + e^{-2\pi j \frac{v}{N}}) \\ &+ \frac{1}{16}(e^{2\pi j \frac{u+v}{N}} + e^{2\pi j \frac{u-v}{N}} + e^{2\pi j \frac{v-u}{N}} + e^{-2\pi j \frac{u+v}{N}}) \\ &= \frac{1}{4} + \frac{1}{4}(\cos\frac{2\pi u}{N} + \cos\frac{2\pi v}{N}) + \frac{1}{8}(\cos\frac{2\pi(u+v)}{N} + \cos\frac{2\pi(u-v)}{N}) \\ &= \frac{1}{4}(1 + \cos\frac{2\pi u}{N} + \cos\frac{2\pi v}{N} + \cos\frac{2\pi u}{N}\cos\frac{2\pi v}{N}) \\ &= \frac{1}{4}(1+\cos\frac{2\pi u}{N})(1+\cos\frac{2\pi v}{N}) = \cos^2\frac{\pi u}{N}\cos^2\frac{\pi v}{N}. \end{aligned}$$

5. Note that $g = h * f$, where

$$h(i,j) = \begin{cases} \frac{1}{\lambda} & \text{if } i = j \leq \lambda - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 \leq u, v \leq N-1$. Then

$$\begin{aligned} DFT(h)(u,v) &= \frac{1}{\lambda N^2}\sum_{k=0}^{\lambda-1} e^{-2\pi j \frac{ku+kv}{N}} \\ &= \begin{cases} \frac{1}{\lambda N^2}\frac{1-e^{-2\pi j \frac{\lambda(u+v)}{N}}}{1-e^{-2\pi j \frac{u+v}{N}}} & \text{if } e^{-2\pi j \frac{u+v}{N}} \neq 1 \\ \frac{1}{N^2} & \text{if } e^{-2\pi j \frac{u+v}{N}} = 1 \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2}\frac{e^{-\pi j \frac{\lambda(u+v)}{N}}(e^{\pi j \frac{\lambda(u+v)}{N}} - e^{-\pi j \frac{\lambda(u+v)}{N}})}{e^{-\pi j \frac{u+v}{N}}(e^{\pi j \frac{u+v}{N}} - e^{-\pi j \frac{u+v}{N}})} & \text{if } u+v \notin N\mathbb{Z} \\ \frac{1}{N^2} & \text{if } u+v \in N\mathbb{Z} \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2} e^{-\pi j \frac{(\lambda-1)(u+v)}{N}} \frac{\sin\frac{\lambda\pi(u+v)}{N}}{\sin\frac{\pi(u+v)}{N}} & \text{if } u+v \notin \{0, N\} \\ \frac{1}{N^2} & \text{if } u+v \in \{0, N\}. \end{cases} \end{aligned}$$

Since $DFT(h * f) = N^2 DFT(h) \odot DFT(f)$,

$$H(u,v) = N^2 DFT(h)(u,v) = \begin{cases} \frac{1}{\lambda} e^{-\pi j \frac{(\lambda-1)(u+v)}{N}} \frac{\sin \frac{\lambda \pi (u+v)}{N}}{\sin \frac{\pi (u+v)}{N}} & \text{if } u+v \notin \{0, N\} \\ 1 & \text{if } u+v \in \{0, N\} \end{cases}.$$