MATH3360: Mathematical Imaging

Assignment 2 Solutions

1. (a) $\int_{\mathbb{R}}[H_0(t)]^2\,dt = \int_0^1 dt = 1.$

   For any $p \in \mathbb{N}\cup\{0\}$ and $n \in \mathbb{Z}\cap[0, 2^p-1]$,

$$\begin{aligned}\int_{\mathbb{R}}[H_{2^p+n}(t)]^2\,dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}}(2^{\frac{p}{2}})^2\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}}(-2^{\frac{p}{2}})^2\,dt\\ &= 2\cdot\frac{1}{2^{p+1}}\cdot 2^p = 1.\end{aligned}$$

   (b) i. Let $m \in \mathbb{N}\setminus\{0\}$. There exists $p \in \mathbb{N}\cup\{0\}$ and $n \in \mathbb{Z}\cap[0,2^p-1]$ such that $m = 2^p+n$. Then

$$\begin{aligned}\langle H_0, H_m\rangle &= \int_{\mathbb{R}} H_0(t)H_{2^p+n}(t)\,dt\\ &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}}(-2^{\frac{p}{2}})\,dt\\ &= \frac{1}{2^{p+1}}\cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}}\cdot(-2^{\frac{p}{2}}) = 0.\end{aligned}$$

   ii. A. Suppose $p_1 = p_2$. Then

$$\begin{aligned}\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t)H_{2^{p_1}+n_2}(t)\,dt\\ &= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}}\cdot 0\,dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n+1}{2^{p_1}}}(-2^{\frac{p_1}{2}})\cdot 0\,dt\\ &\quad+\int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0\cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0\cdot(-2^{\frac{p_1}{2}})\,dt = 0.\end{aligned}$$

   B. Suppose $p_1 < p_2$. Then either

   - $2^{p_2-p_1}n_1 \le n_2 < 2^{p_2-p_1}(n_1+0.5)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$; or
   - $2^{p_2-p_1}(n_1+0.5) \le n_2 < 2^{p_2-p_1}(n_1+1)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$; or
   - $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right) = \varnothing$.

   In any case, H_{m_1} is constant on $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$, and thus denoting the constant by c,

$$\begin{aligned}\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t)H_{2^{p_2}+n_2}(t)\,dt\\ &= c\int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\,dt + c\int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}}(-2^{\frac{p_2}{2}})\,dt\\ &= c[\frac{1}{2^{p_2+1}}\cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}}\cdot(-2^{\frac{p_2}{2}})] = 0.\end{aligned}$$

2. (a) Note that $W_0 = \mathbf{1}_{[0,1)}$ and thus $(W_0)^2 = \mathbf{1}_{[0,1)}$. Recall that for any $n \in \mathbb{N} \cup \{0\}$, W_n is defined by the recursive relation:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(2t) + (-1)^{j + \lfloor \frac{j}{2} \rfloor} W_j(2t-1)$$

for $j \in \mathbb{N} \cup \{0\}$ and $q \in \{0, 1\}$.
Hence for any $n \in \mathbb{N}$, $(W_n)^2 \equiv \mathbf{1}_{[0,1)}$ and thus

$$\int_{\mathbb{R}} [W_n(t)]^2 \, dt = \int_0^1 \, dt = 1.$$

(b) i. Suppose $j_1 = j_2$. Then $m_1 = 2j_1$ and $m_2 = 2j_1 + 1$, and

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1}(t) W_{2j_1+1}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j_1}{2} \rfloor + 1} W_{j_1}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \cdot (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \, dt \\
&= -\int_0^1 [W_{j_1}(u)]^2 \, d(\frac{u}{2}) + \int_0^1 [W_{j_1}(v)]^2 \, d(\frac{v-1}{2}) \\
&= -\frac{1}{2} \|W_{j_1}\|^2 + \frac{1}{2} \|W_{j_1}\|^2 = 0.
\end{aligned}$$

ii. Suppose $j_1 < j_2$. Then

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1+q_1}(t) W_{2j_2+q_2}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor + q_1} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j}{2} \rfloor + q_2} W_{j_2}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2}} W_{j_1}(2t-1) \cdot (-1)^{j_2 + \lfloor \frac{j_2}{2} \rfloor} W_{j_2}(2t-1) \, dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} \cdot \frac{1}{2} \int_0^1 W_{j_1}(u) W_{j_2}(u) \, du \\
&+ (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \cdot \frac{1}{2} \int_0^1 W_{j_1}(v) W_{j_2}(v) \, dv \\
&= \left[(-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} + (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \right] \langle W_{j_1}, W_{j_2} \rangle = 0
\end{aligned}$$

by the induction hypothesis.

Remark. *Recall that $P(m)$ states that*

$$\{W_0, \ldots, W_m\} \textit{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle).$$

Hence even if we have proven $P(m)$ to be true for any $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{W} \textit{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$$

has not been directly proven. The subtle difference is easier to observe if we consider the statements

$$\tilde{P}(m) : \{0, \ldots, m\} \textit{ is finite}$$

and

$$\mathbb{N} \cup \{0\} \text{ is finite,}$$

for which $\tilde{P}(m)$ being true for any $m \in \mathbb{N} \cup \{0\}$ does not imply the truthfulness of the second statement. However, since the orthogonality of $\mathcal{W}$ depends on the orthogonality of pairs of its elements, and each pair of its elements is contained in some $\{W_0 \ldots, W_m\}$, the induction result suffices.

3. (a) For 4×4 images, the transform matrix

$$U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

So, $\hat{A} = UAU = \frac{1}{8}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -j & 1+j & -1 \\ 0 & 1+j & 2 & 1-j \\ 0 & -1 & 1-j & j \end{pmatrix}$

$\hat{B} = UBU = \frac{1}{8} = \begin{pmatrix} 2 & -1-j & 0 & -1+j \\ -1-j & j & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1+j & 1 & 0 & -j \end{pmatrix}$

(b) After discarding 4 smallest entries of the DFT of A, we obtain

$$\hat{A}_{tr} = \frac{1}{8}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1+j & 0 \\ 0 & 1+j & 2 & 1-j \\ 0 & 0 & 1-j & 0 \end{pmatrix}$$

So, $A_{tr} = \mathrm{Re}\big((4U^*)\hat{A}_{tr}(4U^*)\big) = \frac{1}{4}\begin{pmatrix} 5 & -1 & 3 & 1 \\ -1 & 5 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$.

(c) $A * B = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$, $\widehat{A * B} = U(A * B)U = \frac{1}{4}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

Direct computation shows that $\widehat{A * B}(p,q) = 16\hat{A}(p,q)\hat{B}(p,q)$.

4. (a) For any $0 \le p, q \le N-1$,

$$\begin{aligned} iDFT(DFT(f))(p,q) &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{m(k-p)+n(l-q)}{N}} \\ &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\left[\sum_{m=0}^{N-1} e^{2\pi j\frac{m(k-p)}{N}}\right]\left[\sum_{n=0}^{N-1} e^{2\pi j\frac{n(l-q)}{N}}\right] \\ &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\cdot N\mathbf{1}_{N\mathbb{Z}}(k-p)\cdot N\mathbf{1}_{N\mathbb{Z}}(l-q) \\ &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\delta(k-p)\delta(l-q) = f(p,q). \end{aligned}$$

(b) The matrix U used to calculate the DFT of an $N \times N$ matrix is given by

$$U = (U(x,\alpha))_{0 \le x, \alpha \le n}, \text{ where } U(x,\alpha) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{\alpha x}{N}}.$$

Denote by $\vec{u}_\alpha$ the column of U indexed by α. Then for any $0 \le \alpha \le N-1$,

$$\begin{aligned}
\langle \vec{u}_\alpha, \vec{u}_\alpha \rangle &= \sum_{x=0}^{N-1} U(x,\alpha)\overline{U(x,\alpha)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha}{N}} \\
&= N \cdot \frac{1}{N} = 1.
\end{aligned}$$

On the other hand, for any $0 \le \alpha_1, \alpha_2 \le N_1$ such that $\alpha_1 \ne \alpha_2$,

$$\begin{aligned}
\langle \vec{u}_{\alpha_1}, \vec{u}_{\alpha_2} \rangle &= \sum_{x=0}^{N-1} U(x,\alpha_1)\overline{U(x,\alpha_2)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi j \frac{x\alpha_1}{N}} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{x\alpha_2}{N}} \\
&= \frac{1}{N} \sum_{x=0}^{N-1} e^{2\pi j \frac{x(\alpha_1 - \alpha_2)}{N}} \\
&= \frac{1}{N} \cdot N \mathbf{1}_{N\mathbb{Z}}(\alpha_1 - \alpha_2) = 0.
\end{aligned}$$

Hence U is unitary.

(c) Note the the DFT under this new definition is N times the inverse DFT under the original definition. So, this problem reduce to the first question in Tutorial 4.
Hence, we have the formula $\widehat{g * f}(p,q) = N\hat{g}(p,q)\hat{f}(p,q)$.

(d) Let $\tilde{g}$ be the shifted image. Suppose $\tilde{g}(k,l) = g(k-k_0, l-l_0)$. Then,

$$\begin{aligned}
\hat{\tilde{g}}(m,n) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{g}(k,l) e^{2\pi j \frac{mk+nl}{N}} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k-k_0, l-l_0) e^{2\pi j \frac{mk+nl}{N}} \\
&= \frac{1}{N} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l'=-l_0}^{N-1-l_0} g(k',l') e^{2\pi j \frac{mk'+nl'}{N}} e^{2\pi j \frac{mk_0+nl_0}{N}} \\
&= \hat{g}(m,n) e^{2\pi j \frac{mk_0+nl_0}{N}}
\end{aligned}$$

(e) **Coding Assignment:**
Q1:

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recon = (h * U') * freq * (h * U');
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