2017-18 MATH1010
Lecture 9: Differentiation II
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1 Differentiation rules

Proposition 1.1. *If $f(x)$ is a constant function, i.e., $f(x) = c$ for some constant c. Then $f'(x) = 0$.* ■

Proof.

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

□

Proposition 1.2 (The power rule)**.** *If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.* ■

Proof. We will only prove the special case when n is an integer. You can skip the proof. Recall

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

So

$$(x+h)^n - x^n = (x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}).$$

We have

$$\begin{aligned}\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} &= \lim_{h\to 0}((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}) \\ &= x^{n-1} + x^{n-2}x + \cdots xx^{n-2} + x^{n-1} = nx^n.\end{aligned}$$

□

Example 1.1. *Compute*

$$\frac{d}{dx}x^{11}.$$

■

Answer. Applying the power rule, we write

$$\frac{d}{dx}x^{11} = 11x^{10}.$$

Example 1.2. *Compute*

$$\frac{d}{dx}x^{\frac{2}{5}}.$$

■

Answer. Applying the power rule, we write

$$\frac{d}{dx}x^{\frac{2}{5}} = \frac{2}{5}x^{\frac{2}{5}-1} = \frac{2}{5}x^{-\frac{3}{5}}.$$

Example 1.3. *Compute*

$$\frac{d}{dx}\frac{1}{x^3}.$$

■

Answer. Applying the power rule, we write

$$\frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}x^{-3} = -3x^{-4}.$$

Proposition 1.3 (The addition and subtraction rule)**.** *If $f(x)$ and $g(x)$ are differentiable, then so are $f(x) \pm g(x)$ and their derivatives are given by*

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

and

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

■

Proof.

$$\begin{aligned}
\frac{d}{dx}\big(f(x)+g(x)\big) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).
\end{aligned}$$

The other case can be proved similarly. □

Proposition 1.4 (The constant multiple rule)**.** *Let c be a constant. If $f(x)$ is differentiable, then so is $cf(x)$ and its derivative is given by*

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x).$$

■

Proof.

$$\begin{aligned}
\frac{d}{dx}(cf(x)) &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0}\frac{c(f(x+h)-f(x))}{h} \\
&= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= c\frac{d}{dx}f(x).
\end{aligned}$$

□

Example 1.4. *Compute*

$$\frac{d}{dx}\left(x^5 + \frac{1}{x}\right).$$

■

Answer. Write

$$\begin{aligned}\frac{d}{dx}\left(x^5 + \frac{1}{x}\right) &= \frac{d}{dx}x^5 + \frac{d}{dx}x^{-1} \\ &= 5x^4 - x^{-2}.\end{aligned}$$

Example 1.5.

$$\frac{d}{dx}\left(3x^5 - 2x^3 + 1\right).$$

■

Answer. Write

$$\begin{aligned}\frac{d}{dx}\left(3x^5 - 2x^3 + 1\right) &= \frac{d}{dx}(3x^5) + \frac{d}{dx}(-2x^3) + \frac{d}{dx}1 \\ &= 3\frac{d}{dx}x^5 - 2\frac{d}{dx}x^3 + \frac{d}{dx}1 \\ &= 15x^4 - 6x^2.\end{aligned}$$

Example 1.6. *Compute*

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right).$$

■

Answer. Write

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right) &= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7} \\ &= -x^{-4/3} - x^{-1/2} - 7x^{-8}.\end{aligned}$$

2 The product and quotient rule

Warning

$$\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$$

(can you find an example?)

Theorem 2.1 (The product rule)**.** *Suppose $f(x)$ and $g(x)$ are differentiable, then $f(x)g(x)$ is differentiable and*

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

■

Proof. From the limit definition of the derivative, write

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

We then add $0 = -f(x+h)g(x) + f(x+h)g(x)$:

$$\begin{aligned}
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}.
\end{aligned}$$

Now since both $f(x)$ and $g(x)$ are differentiable, they are continuous. Hence

$$\begin{aligned}
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x)\\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}$$

□

Example 2.1. *Let* $f(x) = x^2 + 1$ *and* $g(x) = x^3 - 3x$. *Compute:*

$$\frac{d}{dx}f(x)g(x).$$

■

Answer. Write

$$\begin{aligned}
\frac{d}{dx}f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\
&= (x^2+1)(3x^2-3) + 2x(x^3-3x).
\end{aligned}$$

Expanding this out we have

$$\begin{aligned}
(x^2+1)(3x^2-3) + 2x(x^3-3x) &= 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2\\
&= 5x^4 - 6x^2 - 3,
\end{aligned}$$

Example 2.2. *Suppose* $f(x)$ *is differentiable. Compute*

$$\frac{d}{dx}\left(x^2 f(x)\right).$$

■

Answer. By the product rule

$$\begin{aligned}
\frac{d}{dx}\left(x^2 f(x)\right) &= \left(\frac{d}{dx}x^2\right)f(x) + x^2\left(\frac{d}{dx}f(x)\right)\\
&= 2xf(x) + x^2 f'(x).
\end{aligned}$$

Example 2.3. *Suppose* $f(x)$ *and* $g(x)$ *are differentiable. Given* $f(1) = 1$, $f'(1) = 2$, $g(1) = 3$, $g'(1) = 4$. *Find the value of*

$$\frac{d}{dx}(f(x)g(x))$$

at $x = 1$.

■

Answer. By the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

At $x = 1$, the above is

$$f'(1)g(1) + f(1)g'(1) = 2 \times 3 + 1 = 10.$$

Example 2.4. *Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable. Compute*

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right).$$

■

Answer.

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)h(x)\right) &= \left(f(x)g(x)\right)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}\left(f(x)g(x)\right)\\ &= f(x)g(x)h'(x) + h(x)(f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x))\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).\end{aligned}$$

Theorem 2.2 (The Quotien rule)**.** *If $f(x)$ and $g(x)$ are differentiable, then $\frac{f(x)}{g(x)}$ is differentiable and*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

■

Proof. First note that if we knew how to compute

$$\frac{d}{dx}\frac{1}{g(x)}$$

then we could use the product rule to complete our proof. Write

$$\begin{aligned}\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h}\\ &= \lim_{h\to 0}\frac{g(x) - g(x+h)}{g(x+h)g(x)h}\\ &= \lim_{h\to 0} -\frac{g(x+h) - g(x)}{h}\frac{1}{g(x+h)g(x)}\\ &= -\frac{g'(x)}{g(x)^2}.\end{aligned}$$

Now we can put this together with the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)}\\ &= \frac{-f(x)g'(x) + f'(x)g(x)}{g(x)^2}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

□

Example 2.5. *Compute*

$$\frac{d}{dx}\frac{x^2-1}{x^3+x+1}.$$

■

Answer.

$$\begin{aligned}\frac{d}{dx}\frac{x^2-1}{x^3+x+1} &= \frac{2x(x^3+x+1)-(x^2-1)(3x^2+1)}{(x^3+x+1)^2} \\ &= \frac{-x^4+4x^2+2x+1}{(x^3+x+1)^2}.\end{aligned}$$

Example 2.6. *Compute*

$$\frac{d}{dx}\frac{1-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule. ■

Answer. First, we'll compute the derivative using the quotient rule. Write

$$\frac{d}{dx}\frac{1-x^2}{\sqrt{x}} = \frac{(-2x)\left(\sqrt{x}\right)-(1-x^2)\left(\frac{1}{2}x^{-1/2}\right)}{x}.$$

Second, we'll compute the derivative using the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{1-x^2}{\sqrt{x}} &= \frac{d}{dx}\left(1-x^2\right)x^{-1/2} \\ &= \left(1-x^2\right)\left(\frac{-x^{-3/2}}{2}\right)+(-2x)\left(x^{-1/2}\right).\end{aligned}$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+1}{2x^{3/2}}.$$

3 Differentiation of exponentiation function

Let $f(x)=a^x$. By the definition of derivative

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h-a^x}{h} \\ &= \lim_{h\to 0} a^x\frac{a^h-1}{h} \\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h} \\ &= a^x\cdot\underbrace{(\text{constant})}_{\lim_{h\to 0}\frac{a^h-1}{h}}.\end{aligned}$$

What is $\lim_{h\to 0}\dfrac{a^h-1}{h}$? In below is the table for $a=2$.

h	$(2^h-1)/h$
-1	.5
-0.1	≈0.6700
-0.01	≈0.6910
-0.001	≈0.6929
-0.0001	≈0.6931
-0.00001	≈0.6932

h	$(2^h-1)/h$
1	1
0.1	≈0.7177
0.01	≈0.6956
0.001	≈0.6934
0.0001	≈0.6932
0.00001	≈0.6932

We see that $\lim_{h\to 0} \frac{2^h-1}{h} \approx 0.6932$. From the definition, when a is bigger, the limit is bigger. Strange thing happens when $a = e$:

Proposition 3.1.

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1.$$

(this can be used as the definition of e). ■

Proof.

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$$
$$\frac{e^h-1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots .$$

Hence

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1 + 0 + \cdots = 1.$$

□

Theorem 3.1.

$$\frac{d}{dx} e^x = e^x.$$

■

Proof. From the limit definition of the derivative, write

$$\begin{aligned} \frac{d}{dx} e^x &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} e^x \frac{e^h-1}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h-1}{h} \\ &= e^x. \end{aligned}$$

□

Example 3.1. *Compute* $\frac{d}{dx} x^2 e^x$. ■

Answer. By the product rule

$$\begin{aligned} \frac{d}{dx} x^2 e^x &= \left(\frac{d}{dx} x^2\right) e^x + x^2 \left(\frac{d}{dx} e^x\right) \\ &= 2xe^2 + x^2 e^x. \end{aligned}$$

4 Derivative of trigonometric functions

Theorem 4.1. *1.* $\frac{d}{dx}\sin x = \cos x$

2. $\frac{d}{dx}\cos x = -\sin x$

3. $\frac{d}{dx}\tan x = \sec^2 x$

■

Proof. 1.

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0}\frac{2}{h}\cos(x+\frac{h}{2})\sin(\frac{h}{2})\\ &= \lim_{h\to 0}\cos(x+\frac{h}{2})\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\\ &= \cos x.\end{aligned}$$

2.

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} &= -\lim_{h\to 0}\frac{2}{h}\sin(x+\frac{h}{2})\sin(\frac{h}{2})\\ &= -\lim_{h\to 0}\sin(x+\frac{h}{2})\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\\ &= -\sin x.\end{aligned}$$

3.

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x}\\ &= \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x}.\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.\end{aligned}$$

□