2018 MATH1010F
Lecture 6: The constant e
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The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Motivation

(Source: wikipedia)
Jacob Bernoulli discovered the constant e by studying a question about compound interest.

> An account starts with \$1.00 and pays 100 percent interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be \$2.00. What happens if the interest is computed and credited more frequently during the year?

If the interest is **credited twice** in the year, the interest rate for each 6 months will be 50%, so the initial \$1 is multiplied by 1.5 twice, yielding $\$1.00 \times 1.5^2 = \$2.25$ at the end of the year.
Credit four times Compounding quarterly quarterly yields

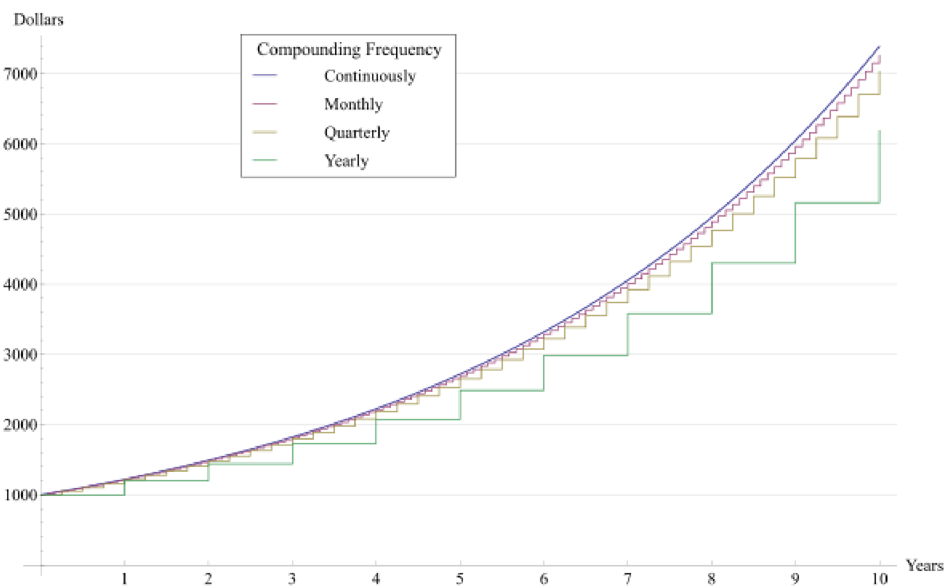
$$\$1.00 \times (1 + 25\%)^4 = \$2.4414...$$

Credit 12 times Compounding monthly yields

$$\$1.00(1 + \frac{1}{12})^{12} = \$2.613035...$$

Credit x times If there are x compounding intervals, the interest for each interval will be $\frac{100}{x}\%$ and the value at the end of the year will be

$$\$1.00(1 + \frac{1}{x})^x.$$



The effect of earning 20% annual interest on an initial \$1,000 investment at various compounding frequencies

x	1	2	5	10	100	1000	10000
$(1+\frac{1}{x})^x$	2	2.25	2.48832	2.59374	2.70481	2.71692	2.71815

The yearly yield increases as x increases. But eventually it approaches a limit. The limit is about 2.7182818... and is denoted by

$$e = \lim_{x\to\infty}(1+\frac{1}{x})^x = 2.7182818...$$

2 Limit involving e

Example 1 Find $\lim_{x\to+\infty}(1+\frac{2}{x})^x$.

$$\lim_{x\to+\infty}(1+\frac{2}{x})^x = \lim_{x\to+\infty}[(1+\frac{1}{x/2})^{x/2}]^2.$$

Let $y = x/2$, as $x \to +\infty, y \to +\infty$. The above is

$$= \left(\lim_{x\to+\infty}(1+\frac{1}{y})^y\right)^2 = e^2.$$

Example 2 Find $\lim_{x\to+\infty}(1+\dfrac{1}{2x-1})^x$.

$$\lim_{x\to+\infty}(1+\frac{1}{2x-1})^x = \lim_{x\to+\infty}(1+\frac{1}{2x-1})^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$

$$= \lim_{x\to+\infty}[(1+\frac{1}{2x-1})^{2x-1}]^{1/2}(1+\frac{1}{2x-1})^{1/2}.$$

Let $y = 2x-1$, as $x \to +\infty, y \to +\infty$.

$$= [\lim_{x\to+\infty}(1+\frac{1}{y})^y]^{1/2}\lim_{x\to+\infty}(1+\frac{1}{2x-1})^{1/2}$$

$$= e^{1/2}\cdot 1 = e^{1/2}.$$

Example 3 Find $\lim_{x\to-\infty}(1+\dfrac{1}{x})^x$.
Let $y = -x$, as $x \to -\infty$, $y \to +\infty$.

$$\lim_{x\to-\infty}(1+\frac{1}{x})^x = \lim_{y\to+\infty}(1-\frac{1}{y})^{-y}$$

$$= \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^y$$

$$= \lim_{y\to+\infty}(1+\frac{1}{y-1})^{y-1}(1+\frac{1}{y-1})$$

$$= e\cdot 1 = e.$$

Remark: From the above, we know that

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e.$$

Example 4 Find $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.
Let $y = \frac{1}{x}$, as $x \to 0$, $y \to \infty$. (As $x \to 0^+$, $y \to +\infty$. As $x \to 0^-$, $y \to -\infty$.)

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}(1+\frac{1}{y})^y = e.$$

3 Properties of e

Recall $0! = 1$, $1! = 1$, $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, $4! = 1 \times 2 \times 3 \times 4$, ... Generally for a non-negative integer n, the factorial of n is defined as

$$n! = 1 \times 2 \times \cdots \times n.$$

We define

$$\exp(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \tag{1}$$

Such expression is called the **power series** of the function $\exp(x)$.

Example 5

$$\exp(1) = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Let count the first 10 terms

$$\sum_{n=0}^{9} \frac{1}{n!} = \frac{98641}{36288} \approx 2.718281525573192 \approx e$$

By the above observation we have

Proposition 1

$$e = \exp(1) = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Proof. proof skipped. □

Proposition 2 *Suppose x, y are real numbers, then*

$$\exp(x + y) = \exp(x)\exp(y).$$

Proof. **The proof can be skipped**

$$\begin{aligned}\exp(x)\exp(y) &= \left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{x^n}{m!}\right)\\ &= \sum_{n,m}\frac{x^n y^m}{n!m!}\\ &= \sum_{k=0}^{\infty}\sum_{n+m=k}\frac{x^n y^m}{n!m!}\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n+m=k}\frac{k!}{n!m!}x^n y^m\\ &= \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!} = \exp(x+y).\end{aligned}$$

Here we use the binomial theorem

$$(x+y)^k = \sum_{n+m=k}\frac{k!}{n!m!}x^n y^m.$$

□

Corollary 3 *For real numbers $x_1, \ldots, x_n$,*

$$\exp(x_1 + x_2 + \cdots + x_n) = \exp(x_1)\exp(x_2)\cdots\exp(x_n).$$

Corollary 4 *For any positive integer n.*

$$\exp(nx) = (\exp(x))^n$$

Proof. Let $x_1 = x_2 = \cdots = x_n = x$. □

Corollary 5 *For positive integer n,*

$$e^n = \exp(n).$$

Proof. Let $x = 1$ in the previous proposition with the fact that $\exp(1) = e$. □

Corollary 6 *Let $x = p/q$ be a fraction. Then*

$$\exp(x) = e^x.$$

Proof.

$$\exp(p/q)^q = \exp(p) = e^p.$$

Hence

$$\exp(p/q) = e^{p/q}.$$

□

From the above, we see that $\exp(x)$ is a natural extension of the function e^x. We can define

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \tag{2}$$

From (2), we can see that e^x grow faster than any polynomials. We have

Proposition 7 *For any k,*

$$\lim_{x \to +\infty} x^k e^{-x} = \lim_{x \to +\infty} \frac{x^k}{e^x} = 0.$$

Proposition 8

$$\lim_{x \to +\infty} p(x) e^{-x} = \lim_{x \to +\infty} \frac{p(x)}{e^x} = 0.$$

Example 6 Find

$$\lim_{x \to +\infty} \frac{e^x - x}{2e^x + 5}.$$

Answer:

$$\lim_{x \to +\infty} \frac{e^x - x}{2e^x + 5} = \lim_{x \to +\infty} \frac{e^{-x}(e^x - x)}{e^{-x}(2e^x + 5)}$$

$$= \lim_{x \to +\infty} \frac{1 - xe^{-x}}{2 + 5e^{-x}} = \frac{\lim_{x \to +\infty}(1 - xe^{-x})}{\lim_{x \to +\infty}(2 + 5e^{-x})} = \frac{1}{2}.$$

Example 7 Find

$$\lim_{x \to 0^+} \frac{1}{x} e^{-1/x}.$$

Answer: Let $y = 1/x$, then

$$\frac{1}{x}e^{-1/x} = ye^{-y}.$$

$$\lim_{x\to 0^+} \frac{1}{x}e^{-1/x} = \lim_{y\to +\infty} ye^{-y} = 0.$$

4 Power Series of sin and cos

Can be skipped.
Recall

$$e^{ix} = \cos x + i\sin x.$$

By (2), the left hand side is

$$\begin{aligned} e^{ix} &= \frac{1}{0!} + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \\ &= \frac{1}{0!} + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots + \cdots \\ &= \left(\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right). \end{aligned}$$

Comparing the real part and the imaginary part, we have

$$\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$