2017-18 MATH1010
Lecture 25: t-method
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1 Substitution $t = \tan x$

Let $t = \tan x$, then we have

$$dx = \frac{dt}{1+t^2}.$$

$$\cos^2 x = \frac{1}{1+t^2}.$$

$$\sin^2 x = \frac{t^2}{1+t^2}.$$

Example 1.1. *Compute*

$$\int \frac{\cos^2 x dx}{1+\sin^2 x}.$$

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Answer. Let $t = \tan x$. Then

$$\begin{aligned}
\int \frac{\cos^2 x dx}{1+\sin^2 x} &= \int \frac{\frac{1}{1+t^2}\frac{dt}{1+t^2}}{1+\frac{t^2}{1+t^2}} \\
&= \int \frac{dt}{(1+t^2)(1+2t^2)} \\
&= \int \left(\frac{2}{1+2t^2} - \frac{1}{1+t^2}\right) dt \qquad \text{(by partial fraction decomposition)} \\
&= \sqrt{2}\int \frac{d(\sqrt{2}t)}{1+(\sqrt{2}t)^2} - \int \frac{dt}{1+t^2} \\
&= \sqrt{2}\tan^{-1}(\sqrt{2}t) - \tan^{-1} t. \\
&= \sqrt{2}\tan^{-1}(\sqrt{2}\tan x) - x.
\end{aligned}$$

Remark: The above method work for expression which can be written as a nice function (e.g. rational function) of $\cos^2 x$, $\sin^2 x$ and

$$\cos x \sin x = \frac{t}{1+t^2}.$$

But it doesn't work, for, say $\int \frac{dx}{1+\sin x}$.

2 t-method

Let
$$t = \tan \frac{x}{2}.$$
Then
$$dx = \frac{2dt}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \tan x = \frac{2t}{1-t^2}.$$

Using this substitution, we can turn a nice function (e.g. rational function) involving sin and cos into another nice function without sin and cos.

Example 2.1. *Compute*
$$\int \frac{dx}{1+\sin x}.$$

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Answer.
$$\begin{aligned}\int \frac{dx}{1+\sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1+\left(\frac{2t}{1+t^2}\right)} \\ &= \int \frac{2dt}{(t+1)^2} \\ &= -\frac{2}{t+1} + C \\ &= -\frac{2}{1+\tan\frac{x}{2}} + C\end{aligned}$$