2017-18 MATH1010
Lecture 23: Trigonometric substitution
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1 More Integrals involving trigonometric functions

Example 1.1. *Evaluate* $\displaystyle\int \tan x \; dx.$ ■

Answer. Rewrite $\tan x$ as $\sin x / \cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x \; dx$, hence $-du = \sin x \; dx$. We can integrate:

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx \\ &= \int \frac{1}{\underbrace{\cos x}_{u}} \underbrace{\sin x \; dx}_{-du} \\ &= \int \frac{-1}{u} \; du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C \\ &= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x \; dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 1.2. *Evaluate* $\displaystyle\int \sec x \; dx.$ ■

Answer.

$$\begin{aligned}\int \sec x \; dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx.\end{aligned}$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x) \; dx$, which is our numerator. Thus:

$$\begin{aligned}&= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |\sec x + \tan x| + C.\end{aligned}$$

We can use similar techniques to those used in Examples 1.1 and 1.2 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 1.1. Antiderivatives of Trigonometric Functions

1. $\int \sin x \; dx = -\cos x + C$
2. $\int \cos x \; dx = \sin x + C$
3. $\int \tan x \; dx = -\ln |\cos x| + C$
4. $\int \csc x \; dx = -\ln |\csc x + \cot x| + C$
5. $\int \sec x \; dx = \ln |\sec x + \tan x| + C$
6. $\int \cot x \; dx = \ln |\sin x| + C$

■

2 Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1 + x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1}5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 2.1. *Evaluate* $\displaystyle\int \frac{1}{25+x^2}\,dx.$ ■

Answer. The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}\ .\end{aligned}$$

Thus

$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1}u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Theorem 2.1. Integrals Involving Inverse Trigonometric Functions
Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right)+C$

■

Example 2.2. *Evaluate the given indefinite integrals.*

$$\int \frac{1}{9+x^2}\,dx, \quad \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad \text{and} \quad \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

■

Answer. Each can be answered using a straightforward application of Theorem 2.1.

$$\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3}+C, \text{ as } a=3.$$

$$\int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx = 10\sec^{-1}10x+C, \text{ as } a=\tfrac{1}{10}.$$

$$\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}}+C, \text{ as } a=\sqrt{5}.$$

Example 2.3. *Evaluate* $\displaystyle\int \frac{1}{x^2-4x+13}\,dx.$ ■

Answer. We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2}-\frac{b^2}{4}+c \\ &= \left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned}\frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13}\\ &= \frac{1}{(x-2)^2+9}\end{aligned}$$

We can now integrate this using the arctangent rule by substituting $u = x-2$. Thus we have

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 2.4. *Evaluate* $\int \frac{4-x}{\sqrt{16-x^2}}\,dx.$ ■

Answer. This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 2.1; the second integral is handled by substitution, with $u = 16-x^2$. We handle each separately.

$\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$

$\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16-x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}}\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\sqrt{u}+C\\ &= -\sqrt{16-x^2}+C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

3 Trigonometric substitution

Example 3.1. *Evaluate* $\int_{-3}^{3} \sqrt{9-x^2}\,dx.$ ■

Answer. We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\ d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\ d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\ d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\ d\theta.
\end{aligned}$$

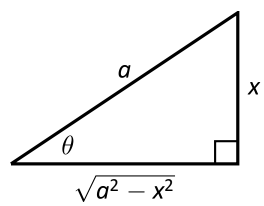
On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

$$\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\ d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\ d\theta \\
&= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.
\end{aligned}$$

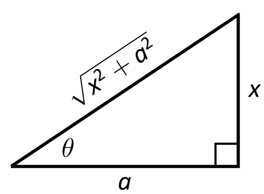
We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ and $\sqrt{x^2 + a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key idea Trigonometric Substitution

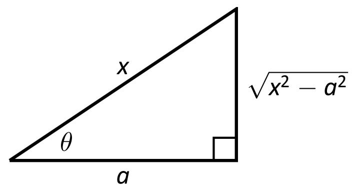
(a) For integrands containing $\sqrt{a^2 - x^2}$
Let $x = a \sin\theta$, $\quad dx = a\cos\theta\, d\theta$
Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.
On this interval, $\cos\theta \ge 0$, so
$\sqrt{a^2 - x^2} = a\cos\theta$



(b) For integrands containing $\sqrt{x^2 + a^2}$:
Let $x = a\tan\theta$, $\quad dx = a\sec^2\theta\, d\theta$
Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.
On this interval, $\sec\theta > 0$, so
$\sqrt{x^2 + a^2} = a\sec\theta$



(c) For integrands containing $\sqrt{x^2 - a^2}$:
Let $x = a\sec\theta$, $\quad dx = a\sec\theta\tan\theta\, d\theta$
Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi/2 < \theta \le \pi$.
We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so
$\sqrt{x^2 - a^2} = a\tan\theta$



Example 3.2. *Evaluate* $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx.$ ■

Answer. Using Key Idea (b), we recognize $a = \sqrt{5}$ and set $x = \sqrt{5}\tan\theta$. This makes $dx = \sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln\left|\sec\theta+\tan\theta\right| + C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea (b) helps. With $x = \sqrt{5}\tan\theta$, we have

$$\tan\theta = \frac{x}{\sqrt{5}} \quad \text{and} \quad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\left|\sec\theta+\tan\theta\right| + C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C.

Example 3.3. *Evaluate* $\int \sqrt{4x^2-1}\ dx$. ■

Answer. We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Key Idea (c), we set $x=\frac{1}{2}\sec\theta$, and hence $dx=\frac{1}{2}\sec\theta\tan\theta\ d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}
\int \sqrt{4x^2-1}\ dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\ dx \\
&= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\ d\theta \\
&= \int \sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\ d\theta \\
&= \int \sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\ d\theta \\
&= \int \frac{1}{2}\tan^2\theta\sec\theta\ d\theta \\
&= \frac{1}{2}\int \Big(\sec^2\theta-1\Big)\sec\theta\ d\theta \\
&= \frac{1}{2}\int \big(\sec^3\theta-\sec\theta\big)\ d\theta.
\end{aligned}$$

We integrated $\sec^3\theta$ in last lecture notes, finding its antiderivatives to be

$$\int \sec^3\theta\ d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}
\int \sqrt{4x^2-1}\ dx &= \frac{1}{2}\int \big(\sec^3\theta-\sec\theta\big)\ d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C \\
&= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.
\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a=1/2$, and $x=\frac{1}{2}\sec\theta$, the reference triangle in Key Idea (c) shows that

$$\tan\theta = \sqrt{x^2-1/4}\Big/(1/2) = 2\sqrt{x^2-1/4} \quad \text{and} \quad \sec\theta = 2x.$$

Thus

$$\frac{1}{4}\Big(\sec\theta\tan\theta - \ln\big|\sec\theta+\tan\theta\big|\Big)+C = \frac{1}{4}\Big(2x\cdot 2\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C$$
$$= \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\;dx = \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.$$

Example 3.4. *Evaluate* $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\;dx.$ ■

Answer. We use Key Idea (a) with $a=2$, $x=2\sin\theta$, $dx=2\cos\theta$ and hence $\sqrt{4-x^2}=2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\;dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\;d\theta\\ &= \int \cot^2\theta\;d\theta\\ &= \int(\csc^2\theta-1)\;d\theta\\ &= -\cot\theta-\theta+C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea (a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta=\sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\;dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right)+C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 3.5. *Evaluate* $\displaystyle\int \frac{1}{x^2+1}\;dx.$ ■

Answer. We know the answer already as $\tan^{-1} x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea (b), let $x = \tan\theta$, $dx = \sec^2\theta\ d\theta$ and note that $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\ dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\ d\theta\\ &= \int 1\ d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and we conclude that $\int \frac{1}{x^2+1}\ dx = \tan^{-1} x + C$.

■

Example 3.6. *Evaluate* $\int \frac{1}{(x^2+6x+10)^2}\ dx.$

Answer. We start by completing the square, then make the substitution $u = x + 3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\ dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\ dx = \int \frac{1}{(u^2+1)^2}\ du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\ d\theta$:

$$\begin{aligned}&= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\ d\theta\\ &= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\ d\theta\\ &= \int \cos^2\theta\ d\theta.\end{aligned}$$

Applying a power reducing formula, we have

$$\begin{aligned}&= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta\\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C.\end{aligned} \tag{1}$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea (b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (1):

$$\begin{aligned}\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.\end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 3.7. *Evaluate* $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx.$ ■

Answer. Using Key Idea (b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx = \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta} 5\sec^2\theta\ d\theta$$
$$= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\ d\theta.$$

We encountered this indefinite integral in Example 3.3 where we found

$$\int \tan^2\theta\sec\theta\ d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$25\int_0^{\pi/4} \tan^2\theta\sec\theta\ d\theta = \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4}$$
$$= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)$$
$$\approx 6.661.$$