2017-18 MATH1010J
Lecture 20: Substitution
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1 Substitution

We motivate this section with an example. Let $f(x) = (x^2+3x-5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2+3x-5)^9 \cdot (2x+3) = (20x+30)(x^2+3x-5)^9.$$

Now consider this: What is $\int (20x+30)(x^2+3x-5)^9 \; dx$? We have the answer in front of us;

$$\int (20x+30)(x^2+3x-5)^9 \; dx = (x^2+3x-5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution*. It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x) \; dx$ as a not–so–complicated integral $\int h(u) \; du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x+30)(x^2+3x-5)^9 \; dx$. Arguably the most "complicated" part of the integrand is $(x^2+3x-5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2+3x-5$. Thus

$$(x^2+3x-5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x+3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}
\int (20x+30)(x^2+3x-5)^9 \ dx &= \int 10(2x+3)(x^2+3x-5)^9 \ dx \\
&= \int 10(\underbrace{x^2+3x-5}_{u})^9 \underbrace{(2x+3) \ dx}_{du} \\
&= \int 10u^9 \ du \\
&= u^{10}+C \quad \text{(replace } u \text{ with } x^2+3x-5\text{)} \\
&= (x^2+3x-5)^{10}+C
\end{aligned}$$

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x) \ dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F\big(u\big)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x) \ dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Theorem 1.1 (Integration by Substitution)**.** *Let F and g be differentiable functions.*

$$\int F'(g(x))g'(x) \ dx = F(g(x)) + C.$$

If $u = g(x)$*, then* $du = g'(x)dx$ *and*

$$\int F'(g(x))g'(x)\ dx = \int F'(u)\ du = F(u) + C = F(g(x)) + C.$$

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In below is a special case $g(x) = ax + b$.

Theorem 1.2 (Substitution With A Linear Function)**.** *Consider* $\int F'(ax+b)\ dx$*, where* $a \neq 0$ *and* b *are constants. Letting* $u = ax+b$ *gives* $du = a \cdot dx$*, leading to the result*

$$\int F'(ax + b)\ dx = \frac{1}{a}F(ax + b) + C.$$

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Example 1.1. *Find*

$$\int (2x + 1)^{2015} dx.$$

Answer. Let $u = g(x) = 2x + 1$, $f(u) = u^{2015}$. Then $du = 2dx$.

$$\begin{aligned}\int (2x + 1)^{2015} dx &= \int u^{2015} \frac{du}{2} \\ &= \frac{u^{2016}}{2 \times 2016} + C \\ &= \frac{(2x + 1)^{2016}}{4032} + C.\end{aligned}$$

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Example 1.2. *Evaluate* $\displaystyle\int \frac{7}{-3x + 1}\ dx.$

Answer. View this a composition of functions $f(g(x))$, where $f(x) = 7/x$ and $u = g(x) = -3x + 1$. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$.

$$\begin{aligned}\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3}\\ &= \frac{-7}{3}\int \frac{du}{u}\\ &= \frac{-7}{3}\ln|u| + C\\ &= -\frac{7}{3}\ln|-3x+1| + C.\end{aligned}$$

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Example 1.3. *Evaluate* $\int xe^{x^2+5}\,dx$

Answer. Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of e^{x^2+5}. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = g(x) = x^2 + 5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int xe^{x^2+5}\,dx &= \int e^{\overbrace{x^2+5}^{u}}\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}e^u\,du\\ &= \frac{1}{2}e^u + C \quad \text{(now replace } u \text{ with } x^2+5\text{)}\\ &= \frac{1}{2}e^{x^2+5} + C.\end{aligned}$$

Thus $\int xe^{x^2+5}\,dx = \frac{1}{2}e^{x^2+5} + C$. We can check our work by evaluating the derivative of the right hand side. ■

Example 1.4. *Evaluate* $\int x^3\sqrt{x^4+1}\ dx$

Answer. Let $u = g(x) = x^4 + 1$, hence $du = 4x^3\,dx$.

$$du = 4x^3\,dx \quad \Rightarrow \quad \frac{1}{4}du = x^3\,dx.$$

We can now substitute.

$$\begin{aligned}\int x^3\sqrt{x^4+1}\ dx &= \int \sqrt{u}\frac{du}{4} \\ &= \frac{1}{6}u^{3/2} + C \\ &= \frac{1}{6}(x^4+1)^{3/2} + C.\end{aligned}$$

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Example 1.5. *Evaluate* $\int \frac{x^3dx}{(x^2+1)^2}$.

Answer. Let $u = x^2 + 1$. Then $du = 2xdu$.

$$\begin{aligned}\int \frac{x^3dx}{(x^2+1)^2} &= \int \frac{x^2du}{2u^2} \\ &= \int \frac{(u-1)du}{2u^2} \qquad \text{(eliminate all the } x\text{)} \\ &= \int \frac{du}{2u} - \int \frac{du}{2u^2} \\ &= \frac{\ln|u|}{2} + \frac{1}{2u} + C \\ &= \frac{\ln(1+x^2)}{2} + \frac{1}{2(1+x^2)} + C\end{aligned}$$

($1 + x^2$ is always positive, so we can remove the absolute sign)

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Remark: Try $\int \frac{x^5dx}{(x^2+1)^2}$. But the method fails for even power for the numerator, e.g. $\int \frac{x^2dx}{(x^2+1)^2}$. We will learn how to integrate this kind of integral later.

Example 1.6. *Evaluate* $\int \frac{e^x dx}{e^x+1}$.

Answer. Let $u = e^x + 1$, $du = e^x dx$.

$$\begin{aligned}\int \frac{e^x dx}{e^x+1} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|e^x+1| + C\end{aligned}$$

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Example 1.7. *Evaluate* $\int x\sqrt{x+3}\ dx$.

Answer. Recognizing the composition of functions, set $u = x+3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\ dx = \int x\sqrt{u}\ du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u - 3 = x$. Thus we can replace x in the integrand with $u-3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\ dx &= \int (u-3)u^{\frac{1}{2}}\ du \\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)\ du \\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem. ■

Example 1.8. *Evaluate* $\int \frac{1}{x\ln x}\ dx$

Answer. This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 1.7 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\ dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\ dx$, which is part of the integrand. Thus:

$$\begin{aligned}\int \frac{1}{x \ln x}\ dx &= \int \underbrace{\frac{1}{\ln x}}_{1/u} \underbrace{\frac{1}{x}\ dx}_{du} \\ &= \int \frac{1}{u}\ du \\ &= \ln |u| + C \\ &= \ln |\ln x| + C.\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct. ■

2 Integration involving substitution of trigonometric functions

Example 2.1. *Evaluate*

$$\int \sin x \cos x dx$$

Answer. Let $u = \sin x$. Then $du = \cos x dx$.

$$\begin{aligned}\int \sin x \cos x dx &= \int u dx \\ &= \frac{u^2}{2} + C \\ &= \frac{1}{2}\sin^2 x + C\end{aligned}$$

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Example 2.2. *Evaluate*

$$\int \sin^3 x dx.$$

Answer. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned}\int \sin^3 x dx &= \int (-\sin^2 x) du \\ &= \int (\cos^2 x - 1) du \\ &= \int (u^2 - 1) du \\ &= \frac{u^3}{3} - u + C \\ &= \frac{\cos^3 x}{3} - \cos x + C.\end{aligned}$$

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Example 2.3. *Evaluate*

$$\int \cos^5 x \sin^2 x dx.$$

Answer. Let $u = \sin x$, $du = \cos x dx$.

$$\begin{aligned}\int \cos^4 x \sin^2 x du &= \int (1 - \sin^2 x)^2 \sin^2 x du \\ &= \int (1 - u^2)^2 u^2 du \\ &= \int (u^2 - 2u^4 + u^6) du \\ &= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C\end{aligned}$$

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Remark: the above method works for integral in the form

$$\int \cos^n x \sin^m x,$$

where one of n, m is odd and the other one is even. The case for n, m having the same parity will be discussed in another lecture

Example 2.4. *Evaluate* $\int \tan x dx$.

Answer. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= \int \frac{-du}{u} \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C\end{aligned}$$

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3 Definite integral using substitution

Proposition 3.1.

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du.$$

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Proof. Let $F(x)$ be the antiderivative of $f(x)$. Then

$$\int f(u(x))u'(x) = \int f(u(x))du(x) = F(u(x)) + C.$$

Hence

$$\int f(u(x))u'(x) = [F(u(x))]_a^b = F(u(a)) - F(u(b)).$$

On the other hand

$$\int_{u(a)}^{u(b)} f(u)du = [F(u)]_{u(a)}^{u(b)} = F(u(a)) - F(u(b)).$$

□

Example 3.1. *Compute*

$$\int_0^1 8x(x^2 + 1)dx.$$

Answer. Let $u = u(x) = x^2 + 1$. Then $du = 2xdx$. When $x = 0$, $u = 0^2 + 1 = 1$. When $x = 1$, $u = 1^2 + 1 = 2$.

$$\begin{aligned}\int_0^1 8x(x^2+1)dx &= \int_1^2 8u\frac{du}{2} \\ &= \left[2u^2\right]_1^2 \\ &= 2 \times 2^2 - 2 \times 1^2 = 6.\end{aligned}$$

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Example 3.2. *Compute*

$$\int_e^{e^2} \frac{1}{x \ln x} dx.$$

Answer. Let $u = \ln x$, $du = \frac{dx}{x}$. When $x = e$, $u = \ln e = 1$. When $x = e^2$, $u = \ln x = 2$.

$$\begin{aligned}\int_e^{e^2} \frac{1}{x \ln x} dx &= \int_1^2 \frac{1}{u} du \\ &= \ln u\Big|_1^2 \\ &= \ln 2 - \ln 1 = \ln 2.\end{aligned}$$

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Example 3.3. *Compute*

$$\int_0^{\pi/4} \tan^3 x \sec^2 x dx.$$

Answer. Let $u(x) = \tan x$, $du = \sec^2 x dx$, $u(0) = 0$ and $u(\frac{\pi}{4}) = 1$.

$$\begin{aligned}\int_0^{\pi/4} \tan^3 x \sec^2 x dx &= \int_0^1 u^3 du \\ &= \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}\end{aligned}$$

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