2018 MATH1010
Lecture 2: Sequence and Series
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The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Sequences

Definition 1 *$a_1, a_2, a_3, \ldots$ is a called a* **sequence** *of real numbers.*

Example 1 (sequence given by formula)

1. Define $a_n = 3n + 1$. Then $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, \ldots$ is a sequence.
2. Define $a_n = 3 \times 2^n$. Then $a_1 = 6, a_2 = 12, a_3 = 24, a_4 = 48, \ldots$ is a sequence.
3. Define $a_n = \frac{1}{n}$. Then $a_1 = \frac{1}{1}, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, a_4 = \frac{1}{4}$ is a sequence.

Definition 2

1. *Let c, d be two fixed real numbers. Let $a_n = c + nd$. The sequence $\{a_n\}$ is called an* **arithmetic sequence***.*
2. *Let c, r be two fixed real numbers. Let $a_n = cr^n$. The sequence $\{a_n\}$ is called a* **geometric sequence***.*

Example 2

1. $a_n = 2 + 3n$ is an arithmetic sequence.
2. $a_n = \frac{3}{2^n}$ is a geometric sequence.

Example 3 (recursive sequence)
$a_1 = 1$, $a_n = 1 + a_{n-1}^2$ for $n \geq 2$. Then $a_1 = 1$, $a_2 = 1 + a_1^2 = 2$, $a_3 = 1 + a_2^2 = 5$, $a_4 = 1 + a_3^2 = 26$, ...

Example 4 (Fibonacci sequence)
$a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Then $a_1 = 1$, $a_2 = 1$, $a_3 = a_2 + a_1 = 2$, $a_4 = a_3 + a_2 = 2 + 1 = 3$, $a_5 = a_4 + a_3 = 3 + 2 = 5$, ...

2 Limit of sequences

Definition 3 *Let $a_1, a_2, \dots,$ be a sequence. If a_n gets closer and closer to a real number L as n gets bigger and bigger, then we say that $\{a_n\}$* **converges** *to L, or L is the* **limit** *of $\{a_n\}$. Denoted by*

$$\lim_{n\to\infty} a_n = L.$$

If no such L exists, then we say that $\{a_n\}$ **diverges**.

Example 5 Let $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$.

Example 6 Let $\{b_n\} = \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots, \}$. Then $\lim_{n\to\infty} b_n = 1$.

Example 7 Let $c_n = \sqrt{n}$, then c_n goes to infinity. We can write $\lim_{n\to\infty} c_n = \infty$.

Example 8 $d_n = (-1)^n$, then d_n diverges.

Example 9 Let k be a constant. Let $a_n = k$. Then $\lim_{n\to\infty} a_n = k$.

Rigorous definition of limit (can be skipped)

We say that

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$, there exists N, such that for $n > N$,

$$|a_n - L| < \varepsilon.$$

Example 10 Use the rigourous definition to show that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then for $n > N$,

$$\left|\frac{1}{n} - 0\right| < \frac{1}{1/\varepsilon} = \varepsilon.$$

Theorem 4 *Let $\{a_n\}$, $\{b_n\}$ be two sequences. Suppose*

$$\lim_{n\to\infty} a_n = A, \lim_{n\to\infty} b_n = B,$$

then we have

1. **Sum rule**: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. **Difference rule**: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. **Constant multiple rule**: $\lim_{n\to\infty} ka_n = kA$, *where* k *is a constant.*
4. **Product rule**: $\lim_{n\to\infty}(a_n b_n) = AB$
5. **Quotient rule**: $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, *if* $B \neq 0$.

Example 11

$$\lim_{n\to\infty}\frac{2}{n} = 2\lim_{n\to\infty} = 2\times 0 = 0.$$

Example 12

$$\lim_{n\to\infty}\frac{n-2}{n} = \lim_{n\to\infty}\left(1-\frac{2}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{2}{n} = 1 - 0 = 1.$$

Example 13

$$\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^2 = \left(\lim_{n\to\infty}\frac{n-2}{n}\right)\left(\lim_{n\to\infty}\frac{n-2}{n}\right) = 1\times 1 = 1.$$

2.1 Limit of rational functions

Example 14 (Technique of finding limit of rational functions)

$$\lim_{n\to\infty}\frac{n^2+1}{n^2-2} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1-\frac{2}{n^2}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)}$$

$$= \frac{1+\lim_{n\to\infty}\frac{1}{n^2}}{1-\lim_{n\to\infty}\frac{2}{n^2}} = \frac{1-0}{1-0} = 1.$$

Example 15

$$\lim_{n\to\infty}\frac{2n^3-3n^2+1}{3n^3+n^2-2} = \lim_{n\to\infty}\frac{(2n^3-3n^2+1)/n^3}{(3n^3+n^2-2)/n^3}$$

$$= \lim_{n\to\infty} \frac{2 - \frac{3}{n} + \frac{1}{n^3}}{3 + \frac{1}{n} - \frac{2}{n^3}} = \frac{2-0+0}{3+0-0} = \frac{2}{3}.$$

Technique: divide the denominator and the numerator by n to the highest power.

Example 16

$$\lim_{n\to\infty} \frac{3n^2-7}{n+8} = \infty.$$

$$\lim_{n\to\infty} \frac{3n^2-7}{n^3+8} = 0.$$

Theorem 5 *Let $p(x)$ and $q(x)$ be two polynomials given by*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_h x^h, \; a_h \neq 0,$$

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k \; b_k \neq 0.$$

Then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \frac{a_h}{b_k} & \text{if } h = k, \\ 0 & \text{if } h < k, \\ \text{diverges} & \text{if } h > k. \end{cases}$$

2.2 Limits involving radicals

Example 17

$$\lim_{n\to\infty} \frac{\sqrt{2n^2+2}}{n+3} = \lim_{n\to\infty} \frac{\sqrt{2+\frac{2}{n^2}}}{1+\frac{3}{n}} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

Example 18

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \lim_{n\to\infty} \frac{n+1-n}{(\sqrt{n+1} + \sqrt{n})} = \lim_{n\to\infty} \frac{1}{(\sqrt{n+1} + \sqrt{n})} = 0.$$

Example 19 Find

$$\lim_{n\to\infty} n^{2/3}((n+1)^{1/3} - n^{1/3}).$$

We use the formula :

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

$$\lim_{n\to\infty} n^{2/3}((n+1)^{1/3} - n^{1/3})$$

$$= \lim_{n\to\infty} \frac{n^{2/3}(n+1)^{1/3} - n^{1/3})((n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3})}{(n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3}}$$

$$= \lim_{n\to\infty} \frac{n^{2/3}}{(n+1)^{2/3} + (n+1)^{1/3}n^{1/3} + n^{2/3}}.$$

Divide the denominator and the numerator by $n^{2/3}$, the above is

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^{2/3} + (1+\frac{1}{n})^{1/3} + 1} = \frac{1}{3}.$$

3 Bounded Monotonic Sequences

Definition 6 *A sequence* $\{a_n\}$ *is* **bounded from above** *(resp.* **bounded from below***) if there exists a number* M *(resp.* m*) such that* $a_n \leq M$ *(resp.* $a_n \geq m$*) for all* n*.*

If $\{a_n\}$ *is bounded from above and below, then it is said to be* **bounded***. Otherwise it is said to be* **unbounded***.*

Example 20 $1, 2, 3, \ldots, n, \ldots$ is unbounded.

Example 21 $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by 1 (or any number greater than 1) and bounded below by $\frac{1}{2}$ (or any number less than $\frac{1}{2}$).

Definition 7 *A sequence* $\{a_n\}$ *is* **monotonic increasing** *(resp. monotonic decreasing) if* $a_n \leq a_{n+1}$ *(resp.* $a_n \geq a_{n+1}$*) for all* n*.*

A sequence is said to be **monotonic** *if it is either monotonic increasing or monotonic decreasing.*

Example 22

1. The sequence $1, 2, 3, \ldots, n, \ldots$ is monotonic increasing.
2. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is monotonic increasing.

3. The sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ is monotonic decreasing
4. The sequence $2, 2, 2, \ldots, 2, \ldots$ is both monotonic increasing and decreasing.
5. The sequence $1, -1, 1, -1, 1, -1, \ldots$ is not monotonic.

Theorem 8 (The Monotonic Sequence Theorem) *If a sequence $\{a_n\}$ monotonic increasing (resp. monotonic decreasing) and bounded from above (resp. below), then the sequence converges.*

Example 23

1. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is monotonic increasing and is bounded above by 1. So $\lim_{n\to\infty} \dfrac{n}{n+1}$ exists.
   In fact, we know that the limit is bounded by 1.
2. The sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ is monotonic decreasing and is bounded below by 0. So $\lim_{n\to\infty} \dfrac{1}{n}$ exists.
   In fact, we know that the limit is 0.

Example 24

$$
\begin{aligned}
a_1 &= \frac{1}{0!} = 1, \\
a_2 &= \frac{1}{0!} + \frac{1}{1!} = 2, \\
a_3 &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = 2.5, \\
&\vdots \\
a_n &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}
\end{aligned}
$$

Obviously the sequence is monotonic increasing. Then

$$a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \leq 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

So a_n converges. In fact to converges to $e = 2.7182818459...$ and is one of the very important constants in mathematics (the other one is π).

A Mysterious formula (can be skipped, just for fun)
$0, 1, \pi, e, i = \sqrt{-1}$ are five important constant in mathematics, the are related by

$$e^{\pi i} + 1 = 0.$$

Example 25 Let $\{a_n\}$ be a sequence defined by

$$a_1 = 1,\ a_{n+1} = 1 + \frac{a_n}{1 + a_n}, \text{for } n \geq 1.$$

Show that $\lim_{n\to\infty} a_n$ exists.

Answer:
Step 1. $\{a_n\}$ is monotonic increasing:
We are going to prove the statement by induction.
Step 1a:

$$a_2 - a_1 = (1 + \frac{a_1}{1 + a_1}) - a_1 = \frac{3}{2} - 1 = \frac{1}{2} > 0.$$

Step 1b: Assume $a_{k+1} \geq a_k$. Then

$$\begin{aligned} a_{k+2} - a_{k+1} &= (1 + \frac{a_{k+1}}{1 + a_{k+1}}) - (1 + \frac{a_k}{1 + a_k}) \\ &= \frac{a_{k+1}}{1 + a_{k+1}} - \frac{a_k}{1 + a_k} \\ &= \frac{a_{k+1} - a_k}{(1 + a_{k+1})(1 + a_k)} \geq 0 \end{aligned}$$

Step 2 $\{a_n\}$ is bounded above by 2:
Obviously all a_n are positive.

$$1 + \frac{a_n}{1 + a_n} \leq 1 + 1 = 2.$$

Hence by the Monotone convergence theorem, $\lim_{n\to\infty} a_n$ exists. Let A be the limit.
Step3: How to find the limit A?

$$a_{n+1} = 1 + \frac{a_n}{1 + a_n}.$$

Then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(1 + \frac{a_n}{1 + a_n}\right) = 1 + \frac{\lim_{n\to\infty} a_n}{1 + \lim_{n\to\infty} a_n}$$

Thus

$$A = 1 + \frac{A}{1+A}$$
$$A^2 + A - 1 = 0.$$

Hence

$$A = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \text{ (rejected because } A \text{ is positive).}$$

Theorem 9 *Suppose* $0 < r < 1$*, then* $\lim_{n\to\infty} r^n = 0$.

Proof. Let $a_n = r^n$. Then $a_{n+1} = ra_n < a_n$ and $a_n > 0$. Hence a_n is a monotonic decreasing sequence bounded below, so it converges to a number A. Next

$$A = \lim_{n\to\infty} r^{n+1} = \lim_{n\to\infty} rr^n = r \lim_{n\to\infty} r^n = rA.$$

Thus $A = 0$. □

4 The Sandwich Theorem

Theorem 10 (The Sandwich Theorem) *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be sequences with* $a_n \le b_n \le c_n$ *for all* n*, and if* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$*. Then* $\lim_{n\to\infty} b_n = L$*.*

Example 26

1.
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$
because
$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}.$$
Let $a_n = -\frac{1}{n}$, $b_n = \frac{(-1)^n}{n}$ and $c_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$. The result follows from the Sandwich theorem.

2.
$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

because

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}.$$

Let $a_n = -\frac{1}{n}$, $b_n = \frac{\cos n}{n}$ and $c_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$. The result follows from the Sandwich theorem.

3. If $|r| < 1$, then

$$\lim_{n\to\infty} r^n = 0$$

as

$$-|r|^n \leq r^n \leq |r|^n.$$

Example 27 Alternate proof of

$$\lim_{n\to\infty} r^n = 0$$

for $0 < r < 1$.
Let $b = \frac{1}{r} - 1$. Then $r = \frac{1}{1+b}$. By the binomial theorem

$$(1+b)^n = 1 + nb + \cdots ... \geq 1 + nb.$$

So

$$0 \leq r^n \leq \frac{1}{(1+b)^n} \leq \frac{1}{1+nb}.$$

Because $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{1+nb} = 0$. By the Sandwich theorem $\lim_{n\to\infty} r^n = 0$.

5 Series

Let $\{a_n\}$ be a sequence. Let

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ a_3 &= a_1 + a_2 + a_3 \\ &\vdots\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

The expression

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is call and **infinite series**, a_n is called the **n-th term** of the series, s_n is called the **n-th partial sum**, the sequence $\{s_n\}$ is called the **sequence of partial sums**. If

$$\lim_{n\to\infty} s_n = L,$$

we say the series **converges** and the sum is L, denoted by

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots = L.$$

Otherwise the series **diverges**.

Theorem 11 *Suppose $|r| < 1$, then the infinite series*

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Proof.

$$s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}.$$

$$\lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1 - \lim_{n\to\infty} r^n}{1-r} = \frac{1}{1-r}.$$

□

We can show that if $|r| > 1$, the infinite series diverges.

Example 28

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges to a constant e.

Example 29 Let

$$a_n = \frac{1}{n(n+1)}.$$

Show that $\sum_{k=1}^{\infty} a_k$ converges and the sum is 1.

Answer

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

$$s_n = a_1 + a_2 + \cdots + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1})$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Then

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{n}) = 1.$$

Example 30 Show that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.
Answer: Because $a_k = \frac{1}{k^2}$ is positive, so s_n is a monotonic increasing sequence. Also for $k \geq 2$,

$$a_k = \frac{1}{k^2} \leq \frac{1}{k(k-1)}$$

$$s_n \leq 1 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \leq 1 + 1.$$

The last step is from the monotone sequence theorem. So the sum converges.
Interesting facts, just for fun, can be skipped In fact, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{60},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^8} = \frac{\pi^8}{9450}.$$

But $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and there is no easy formula for $\sum_{k=1}^{\infty} \frac{1}{k^3}$. There is a famous Riemann Zeta function:

$$\zeta(s) = \sum_{k=1}^{\infty} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

and also an important conjecture called the **Riemann hypothesis**. You can get **one million US dollars** if you solve it.