2017-2018 MATH1010J
Lecture 16: Taylor Series
Charles Li

Warning: Skip the material involving the estimation of error term

Reference: APEX Calculus.
This lecture introduced **Taylor Polynomial** and **Taylor Series**. One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

1 Question related to polynomials

Example 1.1. *Suppose*

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n.$$

Find a_k.

Answer.

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

The steps are given as below.

$$f(c) = a_0.$$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1}.$$

So

$$f'(c) = a_1.$$

$$f''(x) = 2a_2 + 6a_3(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2}.$$

So

$$f''(c) = 2a_2.$$

Similarly

$$f^{(k)}(x) = k!a_k + (2\cdots(k+1))(x-c) + (3\cdots(k+2))(x-c)^2 + \cdots +$$

So

$$f^{(k)}(c) = k!a_k.$$

Or

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

■

Example 1.2. *Suppose $p_n(x)$ is a polynomial with degree n in the form of*

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

Find $p_n(x)$ such that

$$f(c) = p_n(c), f'(c) = p_n'(c), f''(c) = p_n''(c), \ldots,$$

Answer. We can take

$$a_k = \frac{f^{(k)}(c)}{k!}$$

since

$$f^{(k)}(c) = p_n^{(k)}(c) = k!a_k.$$

■

2 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 1, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x-0)+2 = x+2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.

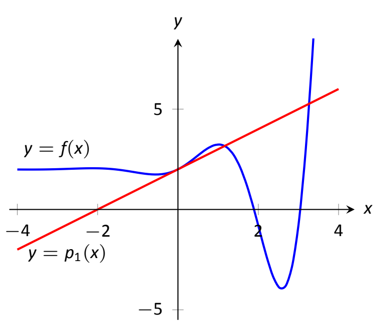

Figure 1: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^{(4)}(0) = -12$
$f''(0) = 2$	$f^{(5)}(0) = -19$

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 1 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

By the discussion in the previous section,

$$p_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 2 + x + x^2.$$

This function is plotted with f in Figure 2.

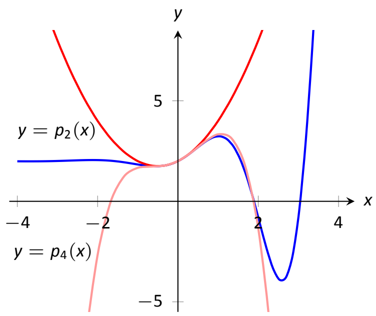

Figure 2: Plotting f, p_2 and p_4.

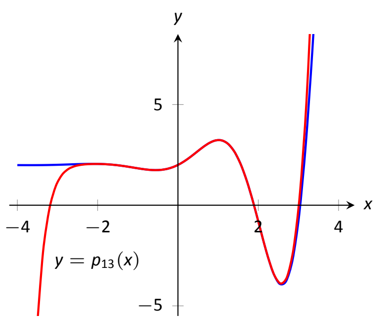

Figure 3: Plotting f and p_{13}.

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 2 also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f. (Using the table in Figure 1, start with $p_4^{(4)}(x) = -12$ and solve the related initial–value problem.)

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 3 shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$\frac{16901x^{13}}{6227020800}+\frac{13x^{12}}{1209600}-\frac{1321x^{11}}{39916800}-\frac{779x^{10}}{1814400}-\frac{359x^{9}}{362880}+\frac{x^{8}}{240}+\frac{139x^{7}}{5040}+\frac{11x^{6}}{360}-\frac{19x^{5}}{120}-\frac{x^{4}}{2}-\frac{x^{3}}{6}+x^{2}+x+2.)$$

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. While we created the above Taylor polynomials by solving initial–value problems, it can be shown that Taylor polynomials follow a general pattern that make their formation much more direct. This is described in the following definition.

Definition 2.1 (Taylor Polynomial, Maclaurin Polynomial)**.** *Let f be a function whose first n derivatives exist at $x = c$.*

1. *The* ***Taylor polynomial of degree n of f centered at $x = c$*** *is*

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. *A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the* ***Maclaurin polynomial of degree n of f*** *is*

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

■

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 2.1. Finding and using Maclaurin polynomials

1. *Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.*
2. *Use $p_5(x)$ to approximate the value of e.*

Answer.

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$.

$$
\begin{array}{lll}
f(x) = e^x & \Rightarrow & f(0) = 1 \\
f'(x) = e^x & \Rightarrow & f'(0) = 1 \\
f''(x) = e^x & \Rightarrow & f''(0) = 1 \\
\vdots & & \vdots \\
f^{(n)}(x) = e^x & \Rightarrow & f^{(n)}(0) = 1
\end{array}
$$

By the definition of the Maclaurin series, we have

$$
\begin{aligned}
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n.
\end{aligned}
$$

2. Using our answer from part 1, we have

$$p_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 4.

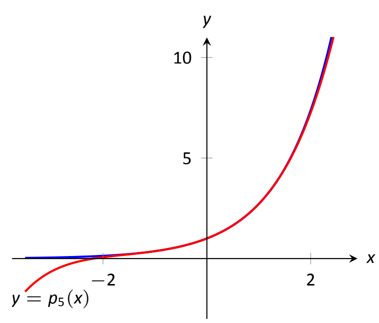

Figure 4: A plot of $f(x) = e^x$ and its 5^{th} degree Maclaurin polynomial $p_5(x)$.

■

Example 2.2. Finding and using Taylor polynomials

1. *Find the n^{th} Taylor polynomial of $y = \ln x$ centered at $x = 1$.*
2. *Use $p_6(x)$ to approximate the value of* $\ln 1.5$.
3. *Use $p_6(x)$ to approximate the value of* $\ln 2$.

Answer.

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown below.

   $f(x) = \ln x \quad \Rightarrow \quad f(1) = 0$
   $f'(x) = 1/x \quad \Rightarrow \quad f'(1) = 1$
   $f''(x) = -1/x^2 \quad \Rightarrow \quad f''(1) = -1$
   $f'''(x) = 2/x^3 \quad \Rightarrow \quad f'''(1) = 2$
   $f^{(4)}(x) = -6/x^4 \quad \Rightarrow \quad f^{(4)}(1) = -6$
   $\vdots \qquad \vdots$
   $f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n} \quad \Rightarrow \quad f^{(n)}(1) = (-1)^{n+1}(n-1)!$

   Using Definition 2.1, we have

   $$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{\,n}(c)}{n!}(x-c)^n \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n. \end{aligned}$$

   Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

2. We can compute $p_6(x)$ using our work above:

   $$p_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6.$$

   Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

   $$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

   This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 5 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

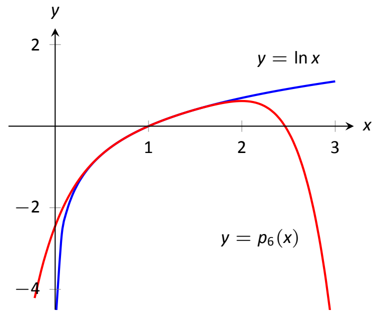

Figure 5: A plot of $y = \ln x$ and its 6^{th} degree Taylor polynomial at $x = 1$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 5 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20^{th} degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 6. We'll soon discuss why this is.

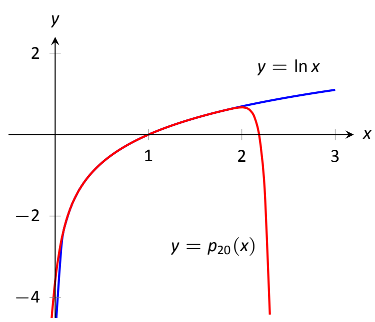

Figure 6: A plot of $y = \ln x$ and its 20^{th} degree Taylor polynomial at $x = 1$.

■

Skip the rest of this section

Theorem 2.1. Skip *[Taylor's Theorem]*

1. *Let f be a function whose $n+1^{th}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that*

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $\left|R_n(x)\right| \leq \dfrac{\max\limits_{z \text{ between } c \text{ and } x} \left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$

■

Proof. Recall

$$f(c) = p_n(c), f'(c) = p_n'(c), f''(c) = p_n''(c), \ldots, f^{(n)}(c) = p_n^{(n)}(c).$$

Let $F(x) = f(x) - p_n(x)$ and $G(x) = (x-c)^{n+1}$. Then

$$F(c) = 0, F'(c) = 0, F''(c) = 0, \ldots, F^{(n)}(c) = 0$$

and

$$G(c) = 0, G'(c) = 0, G''(c) = 0, \ldots, G^{(n)}(c) = 0.$$

We apply Cauchy's mean value theorem to $\frac{F(x)-F(c)}{G(x)-G(c)} = \frac{f(x)-p_n(x)}{(x-c)^n}$ with $a = c$ and $b = x$. Then there exists z_1 between c and x such that

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(0)}{G(x) - G(0)} = \frac{F'(z_1)}{G'(z_1)}.$$

We can use Cauchy's theorem again. There exists z_2 between c and z_1 such that

$$\frac{F'(z_1)}{G'(z_2)} = \frac{F'(z_1) - F'(0)}{G'(z_1) - G'(0)} = \frac{F''(z_1)}{G''(z_2)}.$$

We can repeat the above process n times.

$$\frac{F(x)}{G(x)} = \frac{F'(z_1)}{G'(z_1)} = \frac{F''(z_2)}{G''(z_2)} = \frac{F'''(z_3)}{G'''(z_3)} = \cdots = \frac{F^{(n+1)}(z_{n+1})}{G^{(n+1)}(z_{n+1})},$$

where z_1 is between c and x, z_2 is between c and z_1, etc. Let $z_x = z_{n+1}$. It is between c and x. Note that

$$F^{(n+1)}(z) = f^{(n+1)}(z) \text{ and } G^{(n+1)}(z) = (n+1)!.$$

Hence

$$F(x) = f(x) - p_n(x) = \frac{F^{(n+1)}(z_x)}{(n+1)!}(x-c)^{n+1} = \frac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{n+1}$$

for some z_x between c and x.
The other result follows easily.

□

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 2.2.

Example 2.3. Skip
Finding error bounds of a Taylor polynomial
Use Theorem 2.1 to find error bounds when approximating $\ln 1.5$ *and* $\ln 2$ *with* $p_6(x)$*, the Taylor polynomial of degree 6 of* $f(x) = \ln x$ *centered at* $x = 1$*, as calculated in Example 2.2.*

Answer.

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$.

   The theorem references $\max \left| f^{(n+1)}(z) \right|$. In our situation, this is asking "How big can the 7^{th} derivative of $y = \ln x$ be between 1 and 1.5"? The seventh derivative is $y = -6!/x^7$. The largest value it attains at $x = 1$. Thus we can bound the error as:

$$\begin{aligned} \left|R_6(1.5)\right| &\leq \frac{\max\limits_{z\in[1,1.5]} \left|f^{(7)}(z)\right|}{7!} \left|(1.5-1)^7\right| \\ &\leq \frac{6!}{7!} \cdot \frac{1}{2^7} \\ &\approx 0.0011. \end{aligned}$$

   We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0011. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. The maximum value of the seventh derivative of f on this $[1, 2]$ occurs at $z = 1$.

$$\begin{aligned} \left|R_6(2)\right| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\leq \frac{1}{7}\cdot 1^7 \\ &\approx 0.14286. \end{aligned}$$

This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between $0.61667 - 0.14286 = 0.47381$ and $0.61667 + 0.14286 = 0.75953$. These bounds are not particularly useful.

In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

■

Example 2.4. Skip
Finding sufficiently accurate Taylor polynomials
Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ centered at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Answer. Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm\sin x$ or $\pm\cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\begin{aligned} \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(2-0)^{(n+1)}\right| &\leq 0.001 \\ \frac{1}{(n+1)!}\cdot 2^{(n+1)} &\leq 0.001 \end{aligned}$$

We find an n that satisfies this last inequality with trial–and–error. When $n = 8$, we have $\frac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\frac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given below.

$$\begin{array}{lcl} f(x) = \cos x & \Rightarrow & f(0) = 1 \\ f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\ f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\ f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\ f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\ f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\ f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\ f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\ f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0 \end{array}$$

Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial centered at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned} p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}}{8!}x^8 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \end{aligned}$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure 7 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

■

Example 2.5. Skip
Finding and using Taylor polynomials

1. *Find the degree 4 Taylor polynomial,* $p_4(x)$*, for* $f(x) = \sqrt{x}$ *centered at* $x = 4$*.*
2. *Use* $p_4(x)$ *to approximate* $\sqrt{3}$*.*

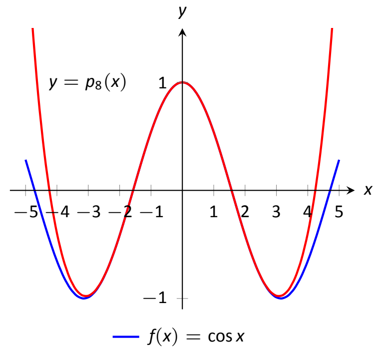

Figure 7: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

3. *Find bounds on the error when approximating* $\sqrt{3}$ *with* $p_4(3)$.

Answer.

1. We begin by evaluating the derivatives of f at $x = 4$.

$$\begin{aligned} f(x) &= \sqrt{x} && \Rightarrow \quad f(4) = 2 \\ f'(x) &= \frac{1}{2\sqrt{x}} && \Rightarrow \quad f'(4) = \frac{1}{4} \\ f''(x) &= \frac{-1}{4x^{3/2}} && \Rightarrow \quad f''(4) = \frac{-1}{32} \\ f'''(x) &= \frac{3}{8x^{5/2}} && \Rightarrow \quad f'''(4) = \frac{3}{256} \\ f^{(4)}(x) &= \frac{-15}{16x^{7/2}} && \Rightarrow \quad f^{(4)}(4) = \frac{-15}{2048} \end{aligned}$$

   These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.

3. $f^{(5)}(x) = \frac{105}{32x^{9/2}}$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ for x between 3 and 4 is at $x = 3$, at about 0.023388. Thus

$$\left|R_4(3)\right| \le \frac{0.023388}{5!}\left|(3-4)^5\right| \approx 0.0001949$$

   This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 8. Note how the two functions are nearly indistinguishable on $(2, 7)$.

■

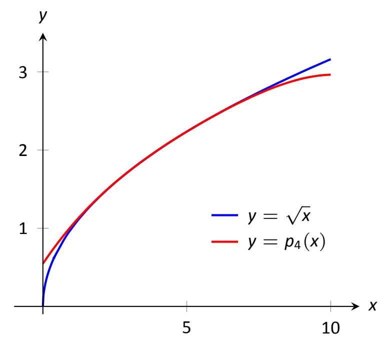

Figure 8: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

3 Taylor Series

In the previous lectures, we showed how certain functions can be represented by a power series function, e.g.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots ,$$

or

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots .$$

In 2, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 3.1 (Taylor and Maclaurin Series)**.** *Let $f(x)$ have derivatives of all orders at $x = c$.*

1. *The* **Taylor Series of $f(x)$, centered at** c *is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. *Setting $c = 0$ gives the* **Maclaurin Series of** $f(x)$*:*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

■

The difference between a Taylor polynomial and a Taylor series is the former is a polynomial, containing only a finite number of terms, whereas the latter is a series, a summation of an infinite set of terms. When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

Example 3.1. The Maclaurin series of $f(x) = \cos x$
Find the Maclaurin series of $f(x) = \cos x$.

Answer. In Example 2.4 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the table shown below: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

$$
\begin{array}{lll}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}
$$

Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

■

Example 3.2. The Taylor series of $f(x) = \ln x$ **at** $x = 1$
Find the Taylor series of $f(x) = \ln x$ *centered at* $x = 1$.

Answer. The table below shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial. The table below shows the derivatives of $\ln x$ evaluated at $x = 1$.

$f(x) = \ln x$	$\Rightarrow$	$f(1) = 0$
$f'(x) = 1/x$	$\Rightarrow$	$f'(1) = 1$
$f''(x) = -1/x^2$	$\Rightarrow$	$f''(1) = -1$
$f'''(x) = 2/x^3$	$\Rightarrow$	$f'''(1) = 2$
$f^{(4)}(x) = -6/x^4$	$\Rightarrow$	$f^{(4)}(1) = -6$
$f^{(5)}(x) = 24/x^5$	$\Rightarrow$	$f^{(5)}(1) = 24$
$\vdots$		$\vdots$
$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$	$\Rightarrow$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centered at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

■

Skip till Example 3.5

It is important to note that Definition 3.1 defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 2.1 states that the error between a function $f(x)$ and its n^{th}–degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$\big|R_n(x)\big| \le \frac{\max\big|f^{(n+1)}(z)\big|}{(n+1)!}\big|(x-c)^{(n+1)}\big|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 3.1 (Function and Taylor Series Equality)**. Skip** *Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 2.1, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim_{n\to\infty} R_n(x) = 0$ for all x in I, then*

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \quad \textit{on } I.$$

■

We demonstrate the use of this theorem in an example.

Example 3.3. Establishing equality of a function and its Taylor series
Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in Example 3.1, for all x.

Answer. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left| f^{\,(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$\left|R_n(x)\right| \leq \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{|x^{n+1}|}{(n+1)!} \leq R_n(x) \leq \frac{|x^{n+1}|}{(n+1)!}. \qquad (1)$$

For any x, $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (1), we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

■

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not the case. In order to properly establish equality, one must use Theorem 3.1.

Example 3.4. Skip
Establishing equality of a function and its Taylor series
Show that $f(x) = \ln x$ is equal to its Taylor series centered at $x = 1$, as found in Example 3.2, for all $x \in (0.5, 2)$.
Remark: *The Taylor series converges on $(0, 2)$ but we can't prove this by our technique in this section.*

Answer. Recall $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$. Given a value $x \in (0.5, 2)$, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\limits_{z \text{ between } x \text{ and } 1.} \left| f^{\,(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{n+1}\right|$$

$$= \left(\max_{z \text{ between } x \text{ and } 1.} \frac{1}{(n+1)z^{n+1}} \right) \left| (x-1)^{n+1} \right|.$$

If $x \geq 1$,

$$\max_{z \text{ between } x \text{ and } 1.} \frac{1}{(n+1)z^{n+1}} = \frac{1}{n+1}.$$

Then

$$\left| R_n(x) \right| \leq \frac{(x-1)^{n+1}}{n+1}.$$

Because $1 > |x-1|$, $\lim_{n\to\infty} \left| R_n(x) \right| = 0$.

For $0.5 < x < 1$,

$$\max_{z \text{ between } x \text{ and } 1.} \frac{1}{(n+1)z^{n+1}} = \frac{1}{(n+1)x^{n+1}}.$$

Then

$$\left| R_n(x) \right| \leq \frac{1}{n+1} \left| 1 - \frac{1}{x} \right|^{n+1}.$$

Again, because

$$\left| 1 - \frac{1}{x} \right| < 1.$$

We can show that $\lim_{n\to\infty} \frac{1}{n+1} \left| 1 - \frac{1}{x} \right|^{n+1} = 0$.

Combining all this,

$$\lim_{n\to\infty} \left| R_n(x) \right| = 0.$$

■

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 3.5. Skip The Binomial Series *Find the Maclaurin series of* $f(x) = (1+x)^k$, $k \neq 0$.

Answer. When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$
\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}
$$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$
1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n + \ldots
$$

The error term

$$
R_{n+1}(x) = \frac{k(k-1)\cdots(k-n)}{(n+1)!}z^{n+1}
$$

for some z between 0 and x. Then

$$
|R_{n+1}(x)| \le \left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}\right| |x|^{n+1}.
$$

We can show that for $|x| < 1$, the RHS tends to 0 as $n \to \infty$.

■

In below are some common Taylor series (**Skip the interval of convergence**)

Function and Series	**First Few Terms**	**Interval of Convergence (skip)**
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$(-1, 1)$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

Theorem 3.2 (Algebra of Power Series)**.** *Let* $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$ *be the Taylor series centered at* $x = c$*. and let* $h(x)$ *be continuous.*

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-c)^n$

2. $$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-c)^n\right) = \sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big)(x-c)^n.$$

3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x) - c\big)^n.$

■

Example 3.6. Combining Taylor series
Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos x$ using table above and Theorem 3.2.

Answer. We can compute all the derivatives of $e^x \cos x$ and compute the Taylor series directly but this method is very slow (try it as an exercise). Instead we use Theorem 3.2

The above table informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Applying Theorem 3.2, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

■

Example 3.7. Creating new Taylor series
Use Theorem 3.2 to create series for $y = \sin(x^2)$.

Answer. We can compute all the derivatives of $\sin(x^2)$ and compute the Taylor series directly but this method is very slow (try it as an exercise). Instead we use Theorem 3.2

Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots .$$

■

Theorem 3.3 (differentiation of the Power Series)**.** *Let* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$. *be the Taylor series centered at* $x = c$.

1. *The Taylor series of* $f'(x)$ *centered at* $x = c$ *is*
$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}.$$
2. *Suppose* $F(x)$ *is an antiderivative of* $f(x)$, *i.e.,* $F'(x) = f(x)$. *Then the Taylor series of* $F(x)$ *centered at* $x = c$ *is*
$$F(0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

■

Example 3.8. *Find the Maclaurin series of* arctan(x).

Answer. We can find the all the derivative of arctan(x) but it will be very slow. (try it!) Instead we use the above theorem.

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

The Taylor series of $\frac{1}{1+x^2}$ is

$$1 - x^2 + x^4 - x^6 + \cdots.$$

Because arctan(x) is an antiderivative of $\frac{1}{1+x^2}$ and $\arctan(0) = 0$. The Taylor series of arctan(x) is therefore

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

■

Interesting formula for π: Let $x = 1$ in the above formula

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

The convergence rate of the above formula for π is very **slow**. To compute π correct to 3 decimal places, we need 4000 terms! Here is a better formula given by Chudnovsky brothers.

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} \frac{(6k)!(13591409 + 545140134k)}{(3k)!(k!)^3(-640320)^{3k}}.$$

Each new term gives about 14 new digits of π!

4 More examples

Example 4.1. *Find the Taylor series of* $\frac{1}{(x-2)(x-3)}$ *centered at* $x = 1$.

Answer.

$$\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$$

$$= \frac{-1}{2-(x-1)} + \frac{1}{1-(x-1)} = \frac{-1}{2}\frac{1}{1-\frac{x-1}{2}} + \frac{1}{1-(x-1)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}(x-1)^n + \sum_{n=0}^{\infty}(x-1)^n$$

$$= \sum_{n=0}^{\infty}(1 - \frac{1}{2^{n+1}})(x-1)^n.$$

■

Example 4.2. *Find the degree 4 Taylor series of* $\sec x$ *and* $\tan x$ *centered at* $x = 0$.

Answer. First of all $\sec x$ is an even function, so the coefficient of the odd degree term of the Taylor series is 0. Hence the degree 4 Taylor series of $\sec x$ centered at $x = 0$ is

$$a_0 + a_2x^2 + a_4x^4.$$

Next

$$\sec x \cos x = 1.$$

Now

$$(a_0+a_2x^2+a_4x^4)(1-\frac{x^2}{2!}+\frac{x^4}{4!}) = a_0+(a_2-\frac{a_0}{2!})x^2+(a_4-\frac{a_2}{2!}+\frac{a_0}{4!})x^4+(\text{terms with degree} > 4).$$

Therefore

$$a_0 = 1$$

$$a_2 - \frac{a_0}{2!} = 0.$$

So

$$a_2 = \frac{1}{2}.$$

$$a_4 - \frac{a_2}{2!} + \frac{a_0}{4!} = 0.$$

So

$$a_4 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}.$$

Hence degree 4 Taylor series of $\sec x$ centered at $x = 0$ is

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}.$$

Next

$$\tan x = \sin x \sec x$$

$$= (x - \frac{x^3}{3!} + (\text{terms with degree} > 4))(1 + \frac{x^2}{2} + \frac{5x^4}{24} + (\text{terms with degree} > 4)).$$

$$x + \frac{x^3}{3} + \text{terms with degree} > 4.$$

Hence degree 4 Taylor series of $\tan x$ centered at $x = 0$ is

$$x + \frac{x^3}{x}.$$

$\blacksquare$