2017-18 MATH1010
Lecture 10: Chain Rule, max/min
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1 The Chain rule

Theorem 1.1 (The Chain Rule)**.** *If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composite function $y = f(g(x))$ is a differentiable function of x whose derivative is given by the product*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or equivalently

$$\frac{dy}{dx} = f'(g(x))g'(x).$$

■

Proof.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{k\to 0}\frac{f(g(x)+k) - f(g(x))}{k}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&\quad (\text{let } k = g(x+h)-g(x)) \\
&= f'(g(x))g'(x).
\end{aligned}$$

□

Example 1.1. *Compute:*

$$\frac{d}{dx}(1+2x)^5.$$

■

Answer. Set $y = f(u) = u^5$ and $u = g(x) = 1 + 2x$. Then $f(g(x)) = (1 + 2x)^5$. Now

$$\frac{dy}{du} = 5u^4 \qquad \text{and} \qquad g'(x) = 2.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(2) = 10(1+2x)^4.$$

Example 1.2. *Compute:*

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}.$$

■

Answer. Let $y = f(u) = \sqrt{u}$, $u = g(x) = 1 + \sqrt{x}$. Then $\sqrt{1+\sqrt{x}} = f(g(x))$.

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}.$$

Example 1.3. *Compute*

$$\frac{d}{dx}e^{\sqrt{x}}.$$

■

Answer. Let $y = f(u) = e^u$, $u = \sqrt{x}$. Then $f(g(x)) = e^{\sqrt{x}}$.

$$\frac{dy}{du} = e^u \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Example 1.4. *Let $g(x)$ be a differentiable function. Find*

$$\frac{d}{dx}(g(x))^n.$$

■

Answer. Let $y = f(u) = u^n$, $u = g(x)$.

$$\frac{dy}{du} = nu^{n-1} \qquad \text{and} \qquad \frac{du}{dx} = g'(x).$$

Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}g'(x) = n(g(x))^{n-1}g'(x).$$

Example 1.5. *Reprove the quotient rule.* ■

Answer.

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx}f(x)g(x)^{-1} \\
&= g(x)^{-1}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)^{-1} \text{ (by the product rule)} \\
&= g(x)^{-1}f'(x) - f(x)g(x)^{-2}g'(x) \text{ (by the previous example with } n = -1) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\end{aligned}$$

Proposition 1.1. *Suppose $y = f(u)$, $u = g(w)$ and $w = h(x)$ are differentiable functions, then $f(g(h(x)))$ is a differentiable function with*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx}$$

or, equivalently

$$\frac{dy}{dx} = f'(g(h(x)))g'(h(x))h'(x).$$

■

Proof. By the Chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and

$$\frac{du}{dx} = \frac{du}{dw}\frac{dw}{dx}.$$

Substitute the second equation into the first one.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx}.$$

□

Remark Similar results can be proved for composite of many functions.

Example 1.6. *Compute*

$$\frac{d}{dx}e^{\sqrt{\frac{x-1}{x+1}}}.$$

■

Answer. Let $y = f(u) = e^u$, $u = g(w) = \sqrt{w}$ and $w = h(x) = \frac{x-1}{x+1}$. Then

$$e^{\sqrt{\frac{x-1}{x+1}}} = f(g(h(x))).$$

Now

$$\frac{dy}{du} = e^u, \ \frac{du}{dw} = \frac{1}{2\sqrt{w}}, \ \frac{dw}{dx} = \frac{2}{(x+1)^2}.$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx}\\ &= e^u\frac{1}{2\sqrt{w}}\frac{2}{(x+1)^2}\\ &= e^{\sqrt{\frac{x-1}{x+1}}}\frac{1}{2\sqrt{\frac{x-1}{x+1}}}\frac{2}{(x+1)^2}\\ &= e^{\sqrt{\frac{x-1}{x+1}}}\frac{1}{(x-1)^{1/2}(x+1)^{3/2}}.\end{aligned}$$

Example 1.7. *Compute*

$$\frac{d}{dx}\sin(\cos(\sqrt{x})).$$

■

Answer. Let $y = f(u) = \sin(u)$, $u = g(w) = \cos(w)$, $w = h(x) = \sqrt{x}$. Then

$$\sin(\cos(\sqrt{x})) = f(g(h(x))).$$

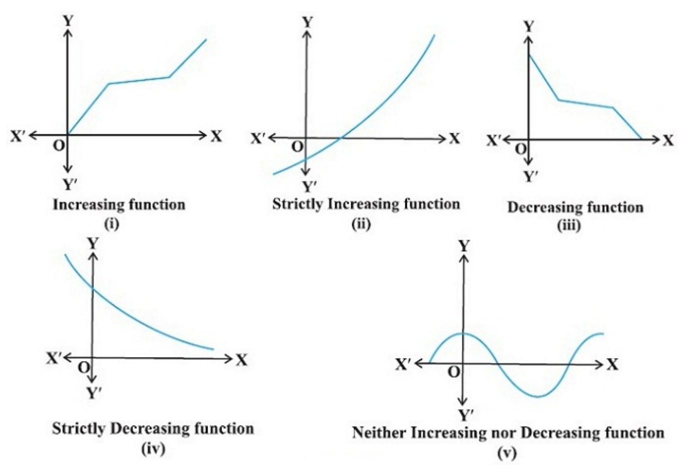
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx} \\ &= \cos(u)(-\sin(w))\frac{1}{2\sqrt{x}} \\ &= -\frac{\cos(\cos(\sqrt{x}))\sin(\sqrt{x})}{2\sqrt{x}}.\end{aligned}$$

2 Increasing/decreasing

Definition 2.1. *Let $f(x)$ be a function defined on an interval. Then*

1. *$f(x)$ is* **increasing** *on the interval if $f(x_2) \geq f(x_1)$ whenever $x_2 \geq x_1$.*
2. *$f(x)$ is* **strictly increasing** *on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.*
3. *$f(x)$ is* **decreasing** *on the interval if $f(x_2) \leq f(x_1)$ whenever $x_2 \geq x_1$.*
4. *$f(x)$ is* **strictly decreasing** *on the interval if $f(x_2) < f(x)$ whenever $x_2 > x_1$.*

■



Theorem 2.1. *Let f be a differentiable function on the interval (a, b).*

1. *If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f(x)$ is an increasing function.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly increasing function.*
3. *If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f(x)$ is a decreasing function.*
4. *If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly decreasing function.*

■

Proof. discussed during class. A rigorous will be given after you learn mean value theorem □

Example 2.1. *Show that $f(x) = e^x - x - 1$ is an increasing function on $[0, \infty)$.* ■

Answer. $f'(x) = e^x - 1 \geq 1 - 1 = 0$. So $f(x)$ is an increasing function.
Interesting observation Because $f(x)$ is an increasing function, $f(x) > f(0) = 0$ for $x > 0$, i.e.

$$e^x > 1 + x, \text{ for } x > 0.$$

Exercise 2.1. *Let k be a positive integer, show that*

$$e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!}$$

is an increasing function on $[0, \infty)$ and hence show that for $x \geq 0$,

$$e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}.$$

■

Procedure to determine the increasing interval and decreasing intervals of f

1. Find all x such that $f'(x) = 0$ or $f'(x)$ is undefined. Mark those points on a number line and divide the line into different intervals.
2. For each intervals (a, b) obtained in the previous step. Pick a number $c \in (a, b)$.
   (a) If $f'(c) > 0$, then $f'(x) > 0$ on (a, b). Hence $f(x)$ is a strict increasing function on (a, b).
   (b) If $f'(c) < 0$, then $f'(x) < 0$ on (a, b). Hence $f(x)$ is a strict decreasing function on (a, b).

Example 2.2. *Find the intervals of increase and decrease of the function*

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

■

Answer.

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1).$$

Now $f'(x) = 0$ if and only if $x = 1$ and $x = -2$. So we have 3 intervals: (i) $(-\infty, -2)$ (ii) $(-2, 1)$ (iii) $(1, \infty)$.

1. For the interval $(-\infty, -2)$, pick $c = -3$, $f'(-3) = 24 > 0$. Therefore $f'(x) > 0$ on this interval and hence is an increasing function on this interval.
2. For the interval $(-2, 1)$, pick $c = 0$, $f'(0) = -6 < 0$. Therefore $f'(x) < 0$ on this interval and hence is a decreasing function on this interval.
3. For the interval $(1, \infty)$, pick $c = 2$, $f'(2) = 24 > 0$. Therefore $f'(x) > 0$ on this interval and hence is an increasing function on this interval.

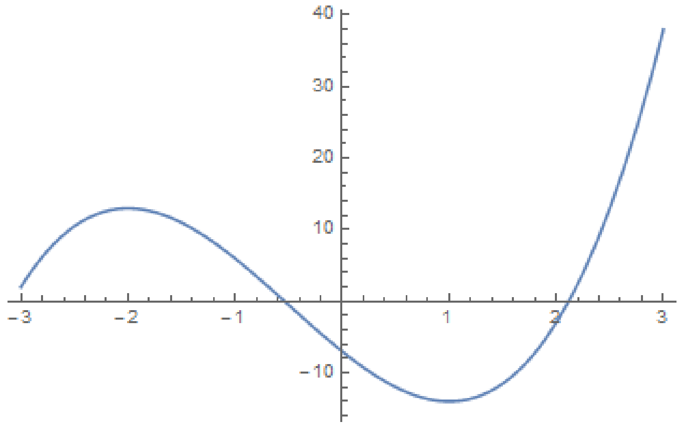


Figure 1: $y = 2x^3 + 3x^2 - 12x - 7$

interval	Test number c	$f'(c)$	conclusion
$x < -2$	-3	$f'(-3) = 24 > 0$	increasing
$-2 < x < 1$	0	$f'(0) = -12 < 0$	decreasing
$-1 < x$	2	$f'(2) = 24 > 0$	increasing

Example 2.3. *Find the intervals of increase and decrease of the function*

$$f(x) = x^7 - 2x^5 + x^3.$$

■

Answer. $f'(x) = 7x^4 - 10x^4 + 3x^2 = x^2(7x^4 - 10x^2 + 3) = 0$. Let $y = x^2$. Consider $7y^2 - 10y + 3$, we have $y = 1$ or $y = \frac{3}{7}$. So the critical points are $x = 0, \pm 1$ and $\pm\sqrt{\frac{3}{7}} \approx \pm 0.654654$.

interval	Test number c	$f'(c)$	conclusion
$x < -1$	-2	$f'(-2) = 300 > 0$	increasing
$-1 < x < -\sqrt{\frac{3}{7}}$	-0.8	$f'(-0.8) = -0.34 < 0$	decreasing
$-\sqrt{\frac{3}{7}} < x < 0$	-0.5	$f'(-0.5) = 0.234 > 0$	increasing
$0 < x < \sqrt{\frac{3}{7}}$	0.5	$f'(0.5) = 0.234 > 0$	increasing
$\sqrt{\frac{3}{7}} < x < 1$	0.8	$f'(0.8) = -0.34 < 0$	decreasing
$1 < x$	2	$f'(2) = 300 > 0$	increasing

Definition 2.2. *Let $f(x)$ be a function with domain D.*

1. *The graph of the function $f(x)$ is said to have a* **relative maximum** *(or* **local maximum***) at $x = c$ if $f(c) \geq f(x)$ for all x in some interval (a, b) containing c.*
2. *The graph of the function $f(x)$ is said to have a* **relative minimum** *(or* **local minimum***) at $x = c$ if $f(c) \leq f(x)$ for all x in some interval (a, b) containing c.*
3. *The relative maxima and minima of f are called its* **relative extrema** *(or* **local extrema***)*

■

Definition 2.3. *Let $f(x)$ be a function with domain D.*

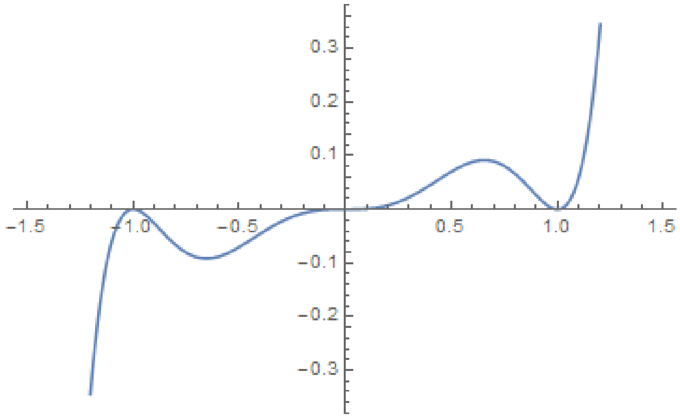
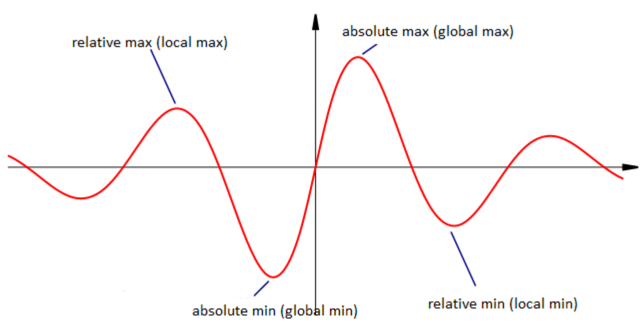


Figure 2: $x^7 - 2x^5 + x^3$

1. *$f(x)$ is said to have a* **absolute maximum** *(or* **global maximum***) at $x = c$ if $f(c) \geq f(x)$ for all x in D.*
2. *$f(x)$ is said to have a* **absolute minimum** *(or* **global minimum***) at $x = c$ if $f(c) \leq f(x)$ for all x in D.*

■



Since a function $f(x)$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$, the only points where $f(x)$ can have a relative extremum are where $f'(x) = 0$ or $f'(x)$ does not exist. Such points are so important that we give them a special names.

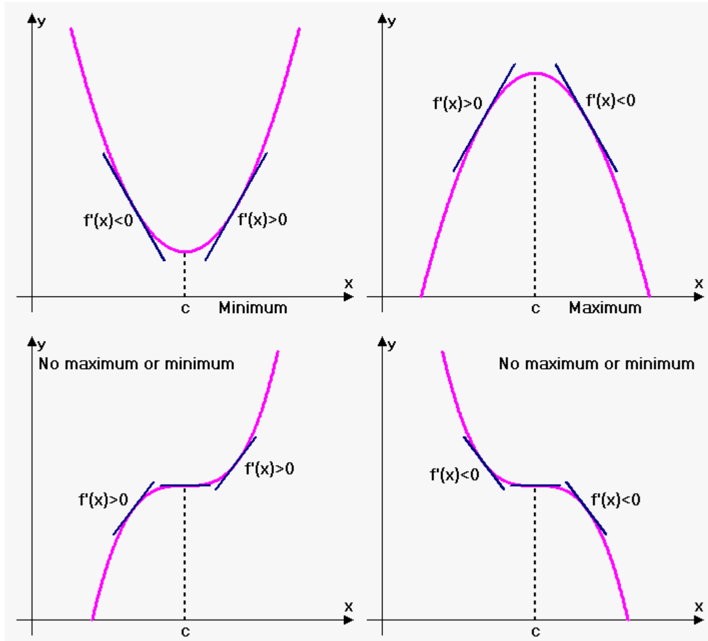
Definition 2.4. *A number c in the domain of $f(x)$ is called a* **critical number** *if either $f'(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a* **critical point** *for $f(x)$.* ■

Theorem 2.2. *Relative extreme can only occur at critical points.* ■

Important remark: Not all critical points correspond to relative extrema.

Theorem 2.3 (The First Derivative Test for Relative Extrema)**.** *Let c be a critical number for $f(x)$. (That is, $f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ does not exist). Then the critical point $(c, f(c))$ is*

1. **Relative maximum** *if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.*
2. **Relative minimum** *if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c.*
3. **not a relative extremum** *if $f'(x)$ has the same sign on both sides of c.*



Property	*Sign of $f'(x)$ to the left of c*	*Sign of $f'(x)$ to the right of c*
Relative maximum	$+$	$-$
Relative minimum	$-$	$+$
Not a relative extremum	$+$	$+$
Not a relative extremum	$-$	$-$

■

Example 2.4. *Let*

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

(See Example 2.2). Find the relative maximum and relative minimum. ■

Answer. Refer to the answer of Example 2.2, $f'(x) = 6x^2 - 6x - 12$. The critical numbers are solutions of $f'(x) = 0$, i.e $x = -1$ and $x = 2$.

x	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$1 < x$
$f'(x)$	$+$	0	$-$	0	$+$

So $(-2, f(-2)) = (-2, 13)$ is a relative maximum and $(1, f(1)) = (1, 14)$ is a relative minimum.

Example 2.5. *Let*

$$f(x) = x^7 - 2x^5 + x^3.$$

(see Example 2.3.) Find the relative maximum and relative minimums. ■

Answer. Refer to the answer of Example 2.3, $f'(x) = 7x^6 - 10x^4 + 3x^2$. The critical numbers are solutions of $f'(x) = 0$, i.e, $x = 0$, $x = \pm 1$ and $x = \pm\sqrt{\frac{3}{7}}$.

x	$x < -1$	$x = -1$	$-1 < x < -\sqrt{\frac{3}{7}}$	$x = -\sqrt{\frac{3}{7}}$	$-\sqrt{\frac{3}{7}} < x < 0$
$f'(x)$	$+$	0	$-$	0	$+$

x	$x = 0$	$0 < x < \sqrt{\frac{3}{7}}$	$x = \sqrt{\frac{3}{7}}$	$\sqrt{\frac{3}{7}} < x < 1$	$x = 1$	$1 < x$
$f'(x)$	0	$+$	0	$-$	0	$+$

So $(-1, f(-1)) = (-1, 0)$, $(\sqrt{\frac{3}{7}}, f(\sqrt{\frac{3}{7}}) \approx (0.655, 0.092)$ are relative maximums. $(-\sqrt{\frac{3}{7}}, f(-\sqrt{\frac{3}{7}})) \approx (-0.655, -0.092)$, $(1, f(1)) = (1, 0)$ are relative minimums. Note that $(0, f(0)$ is neither a relative maximum nor a relative minimum.

3 Second derivative test

Recall: $f''(x) = \frac{d}{dx} f'(x)$.

If $f'(x)$ changes from positive to negative it is decreasing. In this case, $f''(x)$ might be negative, and if in fact $f''(x)$ is negative then $f'(x)$ is definitely decreasing, so there is a relative maximum at the point in question. On the other hand, if $f'(x)$ changes from negative to positive it is increasing. Again, this means that $f''(x)$ might be positive, and if in fact $f''(x)$ is positive then $f'(x)$ is definitely increasing, so there is a relative minimum at the point in question. We summarize this as the *second derivative test.*

Theorem 3.1 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous on an open interval and that $f'(c) = 0$ for some value of c in that interval.*

- *If $f''(c) < 0$, then $f(x)$ has a relative maximum at c.*
- *If $f''(c) > 0$, then $f(x)$ has a relative minimum at c.*
- *If $f''(c) = 0$, then the test is inconclusive. In this case, $f(x)$ may or may not have a relative extremum at $x = c$.*

■

The second derivative test is often the easiest way to identify relative maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 3.1. *Let*

$$f(x) = x^7 - 2x^5 + x^3.$$

(see Example 2.3.) Find the relative maximum and relative minimums by the second derivative test. ■

Answer. Refer to Example 2.3, the critical points are $x = 0$, $x = \pm 1$ and $x = \pm\sqrt{\frac{3}{7}}$. $f'(x) = 7x^6 - 10x^4 + 3x^2$, $f''(x) = 42x^5 - 40x^3 + 6x$.

x	-1	$-\sqrt{\frac{3}{7}}$	0	$\sqrt{\frac{3}{7}}$	1
$f''(x)$	-8	2.2445..	0	$-2.2445...$	8
Property	rel. max.	rel. min.	inconclusive	rel. max.	rel. min.

4 How to find absolute max/min

Theorem 4.1. *Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, then the absolute maximum point and absolute minimum point exist for the graph of f.* ■

Suppose f is a differentiable function on $[a, b]$. If $(c, f(c))$ with $c \in (a, b)$ (i.e., c is not one of the end point) is a absolute maximum or absolute minimum, then it is a relative minimum or relative maximum and hence c is a critical point. The result may not be true if c is one of the end point (i.e. $c = a$ or $c = b$) (why?)

So the possible list of candidates of absolute maximum or minimum are (i) The critical points (ii) the end points a and b. To find the absolute maximum and minimum, we compute the value of f on all the candidates and find the largest one and the smallest one.

Step to find absolute max/min

Suppose f is a differentiable function on $[a, b]$

1. Find all the critical points $f'(x) = 0$ for $x \in (a, b)$. Denote the list of critical points are $c_1, c_2, \ldots,$.
2. Compute $f(a), f(b), f(c_1), f(c_2), \ldots,$ and find the maximum value and the minimum value among them. The maximum value corresponds to the absolute max. The minimum value corresponds to the absolute min.

Example 4.1. *Find the absolute maximum and absolute minimum of the function* $f(x) = x^5 - 80x$ *on the interval* $[-3, 4]$.

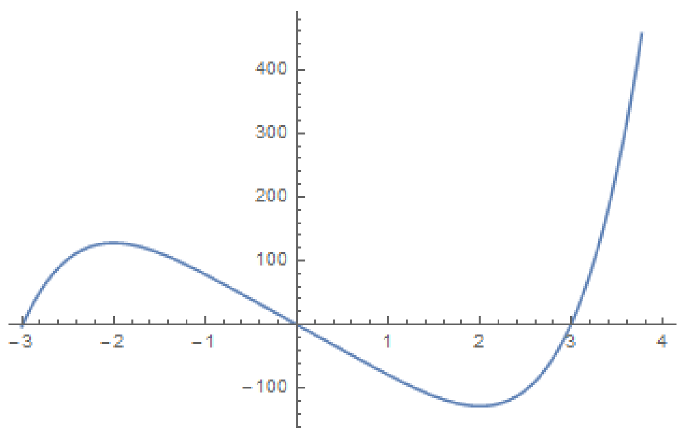


Figure 3: $y = x^5 - 80x$ over $[-3, 4]$

■

Answer. $f'(x) = 5x^4 - 80$. Solve $f'(x) = 0$ for $x \in (-3, 4)$, we have $x = -2$ or 2. Now find the maximum and minimum of the set $f(-2) = 128$, $f(2) = -128$, $f(-3) = -3$, $f(4) = 704$. The absolute minimum is $(2, f(2)) = (2, -128)$ and the absolute maximum is $(4, f(4)) = (4, 704)$.