MATH1010 University Mathematics
Miscellaneous integration

1. Let $a > 0$ and $f(x)$ be a function which is continuous on $[0, a]$.

   (a) Prove that
   $$\int_0^a f(x)dx = \frac{1}{2}\int_0^a (f(x) + f(a-x))\, dx.$$

   (b) Using (a), or otherwise, evaluate the following definite integrals

   (i) $\displaystyle\int_0^1 \frac{x^2 - x + 1}{e^{2x-1} + 1}\, dx$

   (ii) $\displaystyle\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx$

   (iii) $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx$

2. Let $f(x)$ be an even function, i.e. $f(x) = f(-x)$ for all x.

   (a) Show that
   $$\int_{-a}^0 \frac{f(x)}{1 + e^x} dx = \int_0^a \frac{f(x)}{1 + e^{-x}}\, dx.$$

   (b) Show that
   $$\int_{-a}^a \frac{f(x)}{1 + e^x} dx = \int_0^a f(x) dx.$$

   (c) Hence, or otherwise, evaluate
   $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1 + e^x}\, dx.$$

3. (a) Let $f(x)$ be a function such that $f(a - x) = f(x)$ for any real values of x. Prove that
   $$\int_0^a x f(x) dx = \frac{a}{2}\int_0^a f(x) dx.$$

(b) Using the substitution $u = x - \frac{\pi}{2}$, show that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x}\,dx.$$

(c) Evaluate $\displaystyle\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx$.

(d) Use the results of (a) and (c) to evaluate

$$\int_0^{\pi} \frac{x \sin^4 x}{\sin^4 x + \cos^4 x}\,dx.$$

4. (a) Let $0 < a < \frac{\pi}{2}$.

   (i) Prove that $\displaystyle\int_0^a \ln\cos(a-\theta)d\theta = \int_0^a \ln\cos\theta\,d\theta$.

   (ii) Using (a)(i), prove that $\displaystyle\int_0^a \ln(\cos a + \sin a \tan\theta)d\theta = 0$.

   (b) (i) Prove that $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x}\,dx = \frac{\pi \ln 2}{4} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta$.

   (ii) Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x}\,dx$.

5. For positive integer n, let $f_n(x) = x^n e^{-x}$ and define

$$I_n = \int_0^1 f(x)dx.$$

   (a) Prove that $f_n(x)$ is strictly increasing on $(0,1)$.

   (b) Prove that $0 < I_n < \frac{1}{e}$.

   (c) Evaluate I_1.

   (d) Express I_{n+1} in terms of I_n.

   (e) Prove that

$$I_n = n!\left(1 - \frac{1}{e}\sum_{k=0}^{n} \frac{1}{k!}\right).$$

6. For each positive integer n, we define $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$.

(a) Evaluate I_0 and I_1.

(b) Prove that for $n \geq 2$, we have $I_n = \frac{n-1}{n+2} I_{n-2}$.

(c) Prove that for any integer $n \geq 2$,

$$I_n = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\cdots 4\cdot 2}{(n+2)n(n-2)\cdots 7\cdot 5\cdot 3}, & \text{if } n \text{ is odd,} \\ \dfrac{(n-1)(n-3)(n-5)\cdots 3\cdot 1\cdot \pi}{(n+2)n(n-2)\cdots 6\cdot 4\cdot 2}, & \text{if } n \text{ is even.} \end{cases}$$

7. (a) Evaluate the definite integral $\int_0^1 x^4(1-x)^4 dx$.

(b) Evaluate the definite integral $\int_0^1 \frac{x^4(1-x)^4 dx}{1+x^2}$.

(c) By considering the integrals above, show that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}.$$

8. For integer $m, n \geq 0$, let

$$I_{m,n} = \int_0^1 x^m(1-x)^n dx.$$

(a) Prove that for $m \geq 0$ and $n \geq 1$,

$$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}.$$

Hence, show that $I_{m,n} = \dfrac{m!n!}{(m+n+1)!}$.

(b) Show that $\int_0^1 \frac{x^4(1-x)^2}{1+x^2}\, dx = \frac{7}{10} - \ln 2$.

(c) Using (a) and (b), show that

$$\frac{1}{210} \leq \frac{7}{10} - \ln 2 \leq \frac{1}{105}.$$

9. Let $f(x)$ and $g(x)$ be continuous function on $[0,1]$.

   (a) For $x \in [0,1]$, let

   $$H(x) = \int_0^x (f(t))^2 dt \int_0^x (g(t))^2 dt - \left(\int_0^x f(t)g(t) dt \right)^2.$$

   Prove that

   $$H'(x) = \int_0^x (f(x)g(t) - g(x)f(t))^2 dt.$$

   (b) Prove that for any $x \in [0,1]$,

   $$\left(\int_0^x f(t)g(t) dt \right)^2 \leq \int_0^x (f(t))^2 dt \int_0^x (g(t))^2 dt.$$

   (c) Prove that

   $$\left(\int_0^1 x^2 f(x) dx \right)^2 \leq \frac{1}{3} \int_0^1 x^2 (f(x))^2 dx.$$

10. For $n = 0, 1, 2, \ldots$, let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x \, dx$.

    (a) Find I_0 and I_2.

    (b) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ for $n \geq 2$.

    (c) Show that $0 \leq I_n \leq \dfrac{\pi}{4(n+1)}$. (You may use the fact that $0 \leq \tan x \leq \dfrac{4x}{\pi}$ for $0 \leq x \leq \dfrac{\pi}{4}$ without proof.) Hence find $\lim_{n \to \infty} I_n$.

    (d) For $n = 1, 2, 3, \ldots$, let

    $$a_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1}.$$

    (i) Express a_n in terms of I_{2n}.

    (ii) Find $\lim_{n \to \infty} a_n$.

11. Let n be a positive integer.

(a) Show that $\dfrac{1}{1-t^2} = 1 + t^2 + \cdots + t^{2n-2} + \dfrac{t^{2n}}{1-t^2}$ for $t^2 \neq 1$.

(b) For $-1 < x < 1$, show that

(i) $\displaystyle\int_0^x \frac{t}{1-t^2}\,dt = \ln\frac{1}{\sqrt{1-x^2}}$

(ii) $\displaystyle\int_0^x \frac{t^{2n+1}}{1-t^2}\,dt = \ln\frac{1}{\sqrt{1-x^2}} - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{x^{2n}}{2n}\right)$

(c) Show that

$$0 \leq \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k \leq \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}.$$

Hence evaluate $\displaystyle\sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k$.

12. For any real number a such that $|a| \neq 1$, let

$$I(a) = \int_0^{\pi} \ln(1 - 2a\cos x + a^2)dx.$$

(a) Show that $(1-|a|)^2 \leq 1 - 2a\cos x + a^2 \leq (1+|a|)^2$ for any real number x.

(b) Show that

$$2\pi\ln(1-|a|) \leq I(a) \leq 2\pi\ln(1+|a|),$$

and deduce that $\lim\limits_{a\to 0} I(a) = 0$.

(c) Show that $I(a) + I(-a) = I(a^2)$ and $I(a) = I(-a)$. Hence show that $I(a) = \dfrac{1}{2^n}I(a^{2^n})$ for all positive integer n.

(d) Show that

$$I\left(\frac{1}{a}\right) = I(a) - 2\pi\ln|a|.$$

(e) Prove that

$$I(a) = \begin{cases} 0, & \text{if } |a| < 1 \\ 2\pi\ln|a|, & \text{if } |a| > 1 \end{cases}$$

Solution:

1. (a)

$$\begin{aligned}
\int_0^a f(x)\,dx &= \frac{1}{2}\left(\int_0^a f(x)\,dx + \int_0^a f(x)\,dx\right) \\
&= \frac{1}{2}\left(\int_0^a f(x)\,dx - \int_a^0 f(a-y)\,dy\right) \qquad (x = a-y) \\
&= \frac{1}{2}\left(\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx\right) \\
&= \frac{1}{2}\int_0^a (f(x) + f(a-x))\,dx
\end{aligned}$$

(b) (i) Let $f(x) = \dfrac{x^2 - x + 1}{e^{2x-1} + 1}$, we have

$$\begin{aligned}
f(x) + f(1-x) &= \frac{x^2 - x + 1}{e^{2x-1} + 1} + \frac{(1-x)^2 - (1-x) + 1}{e^{2(1-x)-1} + 1} \\
&= \frac{x^2 - x + 1}{e^{2x-1} + 1} + \frac{(1 - 2x + x^2) - (1-x) + 1}{e^{-2x+1} + 1} \\
&= \frac{x^2 - x + 1}{e^{2x-1} + 1} + \frac{(x^2 - x + 1)e^{2x-1}}{1 + e^{2x-1}} \\
&= \frac{(x^2 - x + 1)(1 + e^{2x-1})}{e^{2x-1} + 1} \\
&= x^2 - x + 1
\end{aligned}$$

By (a) we have

$$\begin{aligned}
\int_0^1 f(x)dx &= \frac{1}{2}\int_0^1 (f(x) + f(1-x))\,dx \\
&= \frac{1}{2}\int_0^1 (x^2 - x + 1)dx \\
&= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1 \\
&= \frac{5}{12}
\end{aligned}$$

(ii) Let $f(x) = \dfrac{x \sin x}{1 + \cos^2 x}$, we have

$$\begin{aligned} f(x) + f(\pi - x) &= \frac{x \sin x}{1 + \cos^2 x} + \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} \\ &= \frac{x \sin x}{1 + \cos^2 x} + \frac{(\pi - x)\sin x}{1 + \cos^2 x} \\ &= \frac{\pi \sin x}{1 + \cos^2 x} \end{aligned}$$

By (a) we have

$$\begin{aligned} \int_0^\pi f(x)dx &= \frac{1}{2}\int_0^\pi \left(f(x) + f(\pi - x)\right) dx \\ &= \frac{1}{2}\int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x}\, dx \\ &= -\frac{\pi}{2}\int_0^\pi \frac{d \cos x}{1 + \cos^2 x} \\ &= -\frac{\pi}{2}\left[\tan^{-1} \cos x\right]_0^\pi \\ &= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] \\ &= \frac{\pi^2}{4} \end{aligned}$$

(iii) Let $f(x) = \ln(1 + \tan x)$, we have

$$\begin{aligned} f(x) + f\left(\frac{\pi}{4} - x\right) &= \ln(1 + \tan x) + \ln\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) \\ &= \ln(1 + \tan x) + \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right) \\ &= \ln(1 + \tan x) + \ln\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) \\ &= \ln(1 + \tan x) + \ln\left(\frac{2}{1 + \tan x}\right) \\ &= \ln\left(\frac{2(1 + \tan x)}{1 + \tan x}\right) \\ &= \ln 2 \end{aligned}$$

By (a) we have

$$\begin{aligned}\int_0^{\frac{\pi}{4}} f(x)dx &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(f(x) + f\left(\frac{\pi}{4} - x\right)\right) dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\, dx \\ &= \frac{\pi \ln 2}{8}\end{aligned}$$

2. (a)

$$\begin{aligned}\int_{-a}^{0} \frac{f(x)}{1+e^x}\, dx &= -\int_a^0 \frac{f(-u)}{1+e^{-u}}\, du \qquad (x = -u) \\ &= \int_0^a \frac{f(u)}{1+e^{-u}}\, du \qquad (f(x) \text{ is even}) \\ &= \int_0^a \frac{f(x)}{1+e^{-x}}\, dx\end{aligned}$$

(b)

$$\begin{aligned}\int_{-a}^{a} \frac{f(x)}{1+e^x}\, dx &= \int_{-a}^{0} \frac{f(x)}{1+e^x}\, dx + \int_0^a \frac{f(x)}{1+e^x} dx \\ &= \int_0^a \frac{f(x)}{1+e^{-x}}\, dx + \int_0^a \frac{f(x)}{1+e^x}\, dx \qquad \text{(by (a))} \\ &= \int_0^a \frac{f(x)e^x}{e^x+1}\, dx + \int_0^a \frac{f(x)}{1+e^x}\, dx \\ &= \int_0^a \frac{f(x)(e^x+1)}{e^x+1}\, dx \\ &= \int_0^a f(x)dx.\end{aligned}$$

(c)

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\, dx &= \int_0^{\frac{\pi}{2}} \cos x\, dx \qquad \text{(by (b))} \\ &= -[\sin x]_0^{\frac{\pi}{2}} \\ &= 1\end{aligned}$$

3. (a) Suppose $f(a-x)=f(x)$ for any real values of x.

$$\begin{aligned}
\int_0^a xf(x)dx &= -\int_a^0 (a-u)f(a-u)du \\
&= \int_0^a (a-u)f(u)du \\
&= \int_0^a (a-x)f(x)dx \\
2\int_0^a xf(x)dx &= a\int_0^a f(x)dx \\
\int_0^a xf(x)dx &= \frac{a}{2}\int_0^a f(x)dx
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^4(u+\frac{\pi}{2})}{\sin^4(u+\frac{\pi}{2})+\cos^4(u+\frac{\pi}{2})}du \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{\cos^4 u+\sin^4 u}du \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}dx
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx + \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x+\sin^4 x}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x+\cos^4 x}{\sin^4 x+\cos^4 x}dx \\
&= \int_0^{\frac{\pi}{2}} dx \\
&= \frac{\pi}{2}
\end{aligned}$$

(d) Let $f(x) = \dfrac{\sin^4 x}{\sin^4 x + \cos^4 x}$. Then

$$f(\pi - x) = \frac{\sin^4(\pi - x)}{\sin^4(\pi - x) + \cos^4(\pi - x)} = \frac{\sin^4 x}{\sin^4 x + \cos^4 x} = f(x)$$

By (a) and (c), we have

$$\begin{aligned}\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx \\ &= \frac{\pi^2}{4}\end{aligned}$$

4. (a) (i) Let $u = a - \theta$, $du = -d\theta$.

$$\int_0^a \ln\cos(a-\theta)d\theta = -\int_a^0 \ln\cos u\,du = \int_0^a \ln\cos u\,du = \int_0^a \ln\cos\theta\,d\theta.$$

(ii) By (a)(i),

$$\begin{aligned}\int_0^a (\ln\cos(a-\theta) - \ln\cos\theta)d\theta &= 0 \\ \int_0^a \ln\left(\frac{\cos(a-\theta)}{\cos\theta}\right)d\theta &= 0 \\ \int_0^a \ln\left(\frac{\cos a\cos\theta + \sin a\sin\theta}{\cos\theta}\right)d\theta &= 0 \\ \int_0^a \ln(\cos a + \sin a\tan\theta)d\theta &= 0\end{aligned}$$

(b) (i) Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. When $x = 0$, $\theta = 0$. When

$x = 1$, $\theta = \frac{\pi}{4}$.

$$\begin{aligned}
\int_0^1 \frac{\tan^{-1} x}{1+x}\,dx &= \int_0^{\frac{\pi}{4}} \frac{\theta}{1+\tan\theta}\sec^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{\theta}{1+\tan\theta}\,d\tan\theta \\
&= \int_0^{\frac{\pi}{4}} \theta\,d\ln(1+\tan\theta) \\
&= [\theta\ln(1+\tan\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta \\
&= \frac{\pi\ln 2}{4} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta
\end{aligned}$$

(ii) Taking $a = \frac{\pi}{4}$ in (a)(ii), we have

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\tan\theta\right) &= 0 \\
\int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{\pi}{4}\tan\theta\right) &= 0 \\
\int_0^{\frac{\pi}{4}} \ln\left(\left(1+\frac{\pi}{4}\tan\theta\right) - \ln\sqrt{2}\right)d\theta &= 0 \\
\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta &= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,d\theta \\
&= \frac{\pi\ln 2}{8}
\end{aligned}$$

Now (b)(i) implies that

$$\begin{aligned}
\int_0^1 \frac{\tan^{-1} x}{1+x}\,dx &= \frac{\pi\ln 2}{4} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta \\
&= \frac{\pi\ln 2}{4} - \frac{\pi\ln 2}{8} \\
&= \frac{\pi\ln 2}{8}
\end{aligned}$$

5. (a) For any $n \geq 1$,

$$f_n'(x) = (nx^{n-1} - x^n)e^{-x} = (n-x)x^{n-1}e^{-x} > 0$$

for all $x \in (0,1)$. We see that $f_n(x)$ is strictly increasing on $(0,1)$.

(b) Since $0 = f_n(0) < f_n(x) < f_n(1) = \frac{1}{e}$ for any $x \in (0,1)$, we have

$$\begin{aligned} 0 < \int_0^1 f_n(x)dx &< \int_0^1 \frac{1}{e}\,dx \\ 0 < \qquad I_n \qquad &< \frac{1}{e} \end{aligned}$$

(c)

$$\begin{aligned} I_1 &= \int_0^1 xe^{-x}dx = -\int_0^1 xde^{-x} \\ &= -[xe^{-x}]_0^1 + \int_0^1 e^{-x}dx = -\frac{1}{e} + [-e^{-x}]_0^1 \\ &= 1 - \frac{2}{e} \end{aligned}$$

(d)

$$\begin{aligned} I_{n+1} &= \int_0^1 x^{n+1}e^{-x}dx \\ &= -\int_0^1 x^{n+1}de^{-x} \\ &= -[x^{n+1}e^{-x}]_0^1 + \int_0^1 e^{-x}dx^{n+1} \\ &= -\frac{1}{e} + (n+1)\int_0^1 x^n e^{-x}dx \\ &= (n+1)I_n - \frac{1}{e} \end{aligned}$$

(e)

$$
\begin{aligned}
I_n &= nI_{n-1} - \frac{1}{e} \\
&= n\left((n-1)I_{n-2} - \frac{1}{e}\right) - \frac{1}{e} \\
&= n(n-1)I_{n-2} - \frac{1}{e}(n+1) \\
&= n(n-1)(n-2)I_{n-3} - \frac{1}{e}(n(n-1)+n+1) \\
&\vdots \\
&= n(n-1)(n-2)\cdots 3\cdot 2I_1 - \frac{1}{e}\left(\frac{n!}{2!}+\frac{n!}{3!}+\cdots+n(n-1)+n+1\right) \\
&= n!\left(1-\frac{2}{e}\right) - \frac{n!}{e}\left(\frac{1}{2!}+\cdots+\frac{1}{(n-2)!}+\frac{1}{(n-1)!}+\frac{1}{n!}\right) \\
&= n!\left(1-\frac{1}{e}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{(n-2)!}+\frac{1}{(n-1)!}+\frac{1}{n!}\right)\right) \\
&= n!\left(1-\frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right)
\end{aligned}
$$

6. (a)

$$
\begin{aligned}
I_0 &= \int_0^1 \sqrt{1-x^2}\,dx \\
&= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,d\sin\theta \\
&= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \int_0^1 x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,d(1-x^2) \\
&= -\frac{1}{3}[(1-x^2)^{\frac{3}{2}}]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

(b) For $n \geq 2$,

$$
\begin{aligned}
I_n &= \int_0^1 x^n\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_0^1 x^{n-1}\sqrt{1-x^2}\,d(1-x^2) \\
&= -\frac{1}{3}\int_0^1 x^{n-1}d(1-x^2)^{\frac{3}{2}} \\
&= -\frac{1}{3}\left[\frac{x^{n-1}(1-x^2)^{\frac{3}{2}}}{3}\right]_0^1 + \frac{1}{3}\int_0^1 (1-x^2)^{\frac{3}{2}}dx^{n-1} \\
&= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}dx \\
&= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx \\
&= \frac{n-1}{3}\int_0^1 (x^{n-2}-x^n)\sqrt{1-x^2}\,dx \\
&= \frac{n-1}{3}\,I_{n-2} - \frac{n-1}{3}I_n \\
(n+2)I_n &= (n-1)I_{n-2} \\
I_n &= \frac{n-1}{n+2}\,I_{n-2}
\end{aligned}
$$

(c) When n is odd,

$$\begin{aligned} I_n &= \frac{n-1}{n+2} I_{n-2} \\ &= \frac{(n-1)(n-3)}{(n+2)n} I_{n-4} \\ &= \frac{(n-1)(n-3)(n-5)}{(n+2)n(n-2)} I_{n-6} \\ &\vdots \\ &= \frac{(n-1)(n-3)(n-5)\cdots 4\cdot 2}{(n+2)n(n-2)\cdots 7\cdot 5} I_1 \\ &= \frac{(n-1)(n-3)(n-5)\cdots 4\cdot 2}{(n+2)n(n-2)\cdots 7\cdot 5\cdot 3} \end{aligned}$$

(d) When n is even,

$$\begin{aligned} I_n &= \frac{n-1}{n+2} I_{n-2} \\ &= \frac{(n-1)(n-3)}{(n+2)n} I_{n-4} \\ &= \frac{(n-1)(n-3)(n-5)}{(n+2)n(n-2)} I_{n-6} \\ &\vdots \\ &= \frac{(n-1)(n-3)(n-5)\cdots 3\cdot 1}{(n+2)n(n-2)\cdots 6\cdot 4} I_0 \\ &= \frac{(n-1)(n-3)(n-5)\cdots 3\cdot 1\pi}{(n+2)n(n-2)\cdots 6\cdot 4\cdot 2} \end{aligned}$$

7. (a)

$$\begin{aligned} \int_0^1 x^4(1-x)^4 dx &= \int_0^1 (x^8 - 4x^7 + 6x^6 - 4x^5 + x^4) dx \\ &= \left[\frac{x^9}{9} - \frac{x^8}{2} + \frac{6x^7}{7} - \frac{2x^6}{3} + \frac{x^5}{5} \right]_0^1 \\ &= \frac{1}{630}. \end{aligned}$$

(b)

$$\begin{aligned}\int_0^1 \frac{x^4(1-x^4)dx}{1+x^2} &= \int_0^1 \left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx \\ &= \left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x-4\tan^{-1}x\right]_0^1 \\ &= \frac{22}{7}-\pi.\end{aligned}$$

(c) Since $\frac{x^4(1-x)^4}{2} < \frac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$ on $(0,1)$, we have

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4dx}{2} &< \int_0^1 \frac{x^4(1-x)^4dx}{1+x^2} &< \int_0^1 x^4(1-x)^4dx \\ \frac{1}{1260} &< \frac{22}{7}-\pi &< \frac{1}{630}\end{aligned}$$

8. (a) For $m \geq 0$ and $n \geq 1$, we have

$$\begin{aligned}I_{m,n} &= \int_0^1 x^m(1-x)^n dx \\ &= \frac{1}{m+1}\int_0^1 (1-x)^n dx^{m+1} \\ &= \left[\frac{x^{m+1}(1-x)^n}{m+1}\right]_0^1 - \frac{1}{m+1}\int_0^1 x^{m+1}d(1-x)^n \\ &= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx \\ &= \frac{n}{m+1} I_{m+1,n-1}\end{aligned}$$

Hence

$$\begin{aligned} I_{m,n} &= \frac{n}{m+1} I_{m+1,n-1} \\ &= \frac{n(n-1)}{(m+1)(m+2)} I_{m+2,n-2} \\ &\vdots \\ &= \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{(m+1)(m+2)\cdots(m+n-1)(m+n)} I_{m+n,0} \\ &= \frac{n!}{(m+1)(m+2)\cdots(m+n-1)(m+n)} \int_0^1 x^{m+n} dx \\ &= \frac{m!n!}{(m+n)!} \left[\frac{x^{m+n+1}}{m+n+1}\right]_0^1 \\ &= \frac{m!n!}{(m+n+1)!} \end{aligned}$$

(b)

$$\begin{aligned} \int_0^1 \frac{x^4(1-x)^2}{1+x^2}\, dx &= \int_0^1 \frac{x^6-2x^5+x^4}{1+x^2}\, dx \\ &= \int_0^1 \left(x^4 - 2x^3 + 2x - \frac{2x}{1+x^2}\right) dx \\ &= \left[\frac{x^5}{5} - \frac{x^4}{2} + x^2\right]_0^1 - \int_0^1 \frac{d(1+x^2)}{1+x^2} \\ &= \frac{1}{5} - \frac{1}{2} + 1 - [\ln(1+x^2)]_0^1 \\ &= \frac{7}{10} - \ln 2 \end{aligned}$$

(c) First note that

$$\int_0^1 x^4(1-x)^2 dx = I_{4,2} = \frac{4!2!}{7!} = \frac{1}{105}$$

Since $\dfrac{x^4(1-x)^2}{2} < \dfrac{x^4(1-x)^2}{1+x^2} < x^4(1-x)^2$ on $(0,1)$, we have

$$\begin{array}{ccccc} \displaystyle\int_0^1 \frac{x^4(1-x)^2dx}{2} & < & \displaystyle\int_0^1 \frac{x^4(1-x)^2dx}{1+x^2} & < & \displaystyle\int_0^1 x^4(1-x)^2dx \\ \dfrac{1}{210} & < & \dfrac{22}{7}-\pi & < & \dfrac{1}{150} \end{array}$$

9. (a)

$$\begin{aligned} & H'(x) \\ = \ & (f(x))^2\int_0^x (g(t))^2dt + (g(x))^2\int_0^x (f(t))^2dt - 2f(x)g(x)\int_0^x f(t)g(t)dt \\ = \ & \int_0^x \big((f(x))^2(g(t))^2 + (g(x))^2(f(t))^2 - 2f(x)g(x)f(t)g(t)\big)\,dt \\ = \ & \int_0^x (f(x))(g(t)) - g(x)f(t))^2\,dt \end{aligned}$$

(b) Since $H(0)=0$ and $H'(x)\geq 0$ by (a), we have for any $x\in[0,1]$,

$$\begin{aligned} H(x) &\geq 0 \\ \int_0^x (f(t))^2dt\int_0^x (g(t))^2dt - \left(\int_0^x f(t)g(t)dt\right)^2 &\geq 0 \\ \left(\int_0^x f(t)g(t)dt\right)^2 &\leq \int_0^x (f(t))^2dt\int_0^x (g(t))^2dt \end{aligned}$$

(c) By (b), we have

$$\begin{aligned} \left(\int_0^1 x(xf(x))dx\right)^2 &\leq \int_0^1 x^2dx\int_0^1 (xf(x))^2dx \\ \left(\int_0^1 x^2f(x)dt\right)^2 &\leq \frac{1}{3}\int_0^1 x^2(f(x))^2dx \end{aligned}$$

10. (a)

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4} \\ I_2 &= \int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)dx = [\tan x - x]_0^{\frac{\pi}{4}} = 1-\frac{\pi}{4} \end{aligned}$$

(b) For $n \geq 2$,

$$\begin{aligned} I_n + I_{n-2} &= \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2} x) dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1) dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x d \tan x \\ &= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{n-1} \end{aligned}$$

(c) Since $0 \leq \tan x \leq \frac{4x}{\pi} \leq 1$ for any $0 \leq x \leq \frac{\pi}{4}$, we have

$$\begin{aligned} 0 \leq \; I_n \; &\leq \int_0^{\frac{\pi}{4}} \frac{4^n x^n}{\pi^n} \, dx \\ 0 \leq \; I_n \; &\leq \left[\frac{4^n x^{n+1}}{(n+1)\pi^n} \right]_0^{\frac{\pi}{4}} \\ 0 \leq \; I_n \; &\leq \frac{4^n}{(n+1)\pi^n} \left(\frac{\pi}{4} \right)^{n+1} \\ 0 \leq \; I_n \; &\leq \frac{\pi}{4(n+1)} \end{aligned}$$

Since $\lim_{n \to \infty} \frac{\pi}{4(n+1)} = 0$, it follows by squeeze theorem that

$$\lim_{n \to \infty} I_n = 0$$

(d) (i)

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \\ &= 1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^{n-1}}{2n-1} \\ &= (I_0 + I_2) - (I_2 + I_4) + (I_4 + I_6) + \cdots + (-1)^{n-1}(I_{2n-2} + I_{2n}) \\ &= I_0 + (-1)^{n-1} I_{2n} \end{aligned}$$

(ii)

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (I_0 + (-1)^{n-1} I_{2n}) \\ &= I_0 \\ &= \frac{\pi}{4} \end{aligned}$$

11. (a) For $t^2 \neq 1$,

$$1 + t^2 + \cdots + t^{2n-2} = \frac{1-(t^2)^n}{1-t^2} = \frac{1}{1-t^2} - \frac{t^{2n}}{1-t^2}.$$

(b) (i)

$$\begin{aligned} \int_0^x \frac{t}{1-t^2}\,dt &= -\frac{1}{2}\int_0^x \frac{1}{1-t^2}\,d(1-t^2) \\ &= -\frac{1}{2}[\ln(1-t^2)]_0^x \\ &= \frac{1}{2}\ln\frac{1}{1-x^2} \\ &= \ln\frac{1}{\sqrt{1-x^2}} \end{aligned}$$

(ii)

$$\begin{aligned}
\frac{1}{1-t^2} &= 1+t^2+\cdots+t^{2n-2}+\frac{t^{2n}}{1-t^2}\\
\frac{t}{1-t^2} &= t+t^3+\cdots+t^{2n-1}+\frac{t^{2n+1}}{1-t^2}\\
\int_0^x \frac{t}{1-t^2}\,dt &= \int_0^x (t+t^3+\cdots+t^{2n-1})dt+\int_0^x \frac{t^{2n+1}}{1-t^2}\,dt\\
\ln\frac{1}{\sqrt{1-x^2}} &= \frac{x^2}{2}+\frac{x^4}{4}+\cdots+\frac{x^{2n}}{2n}+\int_0^x \frac{t^{2n+1}}{1-t^2}\,dx\\
\int_0^x \frac{t^{2n+1}}{1-t^2}\,dx &= \ln\frac{1}{\sqrt{1-x^2}}-\left(\frac{x^2}{2}+\frac{x^4}{4}+\cdots+\frac{x^{2n}}{2n}\right)
\end{aligned}$$

(c) Putting $x^2=\dfrac{8}{9}$ in (b)(ii), we have

$$\begin{aligned}
\int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dx &= \ln\frac{1}{\sqrt{1-(\frac{8}{9})^2}}-\left(\frac{(\frac{8}{9})}{2}+\frac{(\frac{8}{9})^2}{4}+\cdots+\frac{(\frac{8}{9})^n}{2n}\right)\\
&= \ln 3-\sum_{k=1}^{n}\frac{1}{2k}\left(\frac{8}{9}\right)^k.
\end{aligned}$$

Now for $0\le t\le\sqrt{\dfrac{8}{9}}$, we have

$$0\le\frac{t^{2n+1}}{1-t^2}\le\frac{t^{2n+1}}{1-\frac{8}{9}}\le 9t^{2n+1}.$$

Thus

$$\begin{aligned}
0\le\int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dt &\le 9\int_0^{\sqrt{\frac{8}{9}}} t^{2n+1}dt\\
&= \left[\frac{9t^{2n+2}}{2n+2}\right]_0^{\sqrt{\frac{8}{9}}}\\
&= \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}
\end{aligned}$$

Hence

$$0 \leq \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k \leq \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}.$$

Since $\lim_{n\to\infty} \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1} = 0$, by squeeze theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\left(\ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k\right) &= 0 \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k &= \ln 3
\end{aligned}$$

12. (a) First

$$-2|a| \leq -2a\cos x \leq 2|a|,$$

and we have

$$(1-|a|)^2 \leq 1 - 2a\cos x + a^2 \leq (1+|a|)^2.$$

(b) Using the inequalities derived in (a),

$$\int_0^\pi (1-|a|)^2 dx \leq \int_0^\pi \ln(1-2a\cos x + a^2)dx \leq \int_0^\pi (1+|a|)^2 dx,$$

that is

$$\pi(1-|a|)^2 \leq I(a) \leq \pi(1+|a|)^2.$$

As $\lim_{a\to 0}(1 \pm |a|)^2 = 0$, by the sandwich theorem, we get

$$\lim_{a\to 0} I(a) = 0.$$

(c) We first have

$$\begin{aligned}
& I(a)+I(-a) \\
= & \int_0^\pi \ln(1-2a\cos x+a^2)dx+\int_0^\pi \ln(1+2a\cos x+a^2)dx \\
= & \int_0^\pi (\ln(1-2a\cos x+a^2)+\ln(1+2a\cos x+a^2)dx \\
= & \int_0^\pi \ln((1-2a\cos x+a^2)(1+2a\cos x+a^2))dx \\
= & \int_0^\pi \ln(1+2a^2(1-2\cos^2 x)+a^4)dx \\
= & \int_0^\pi \ln(1-2a^2\cos 2x+a^4)dx \\
= & \frac{1}{2}\int_0^{2\pi} \ln(1-2a^2\cos t+a^4)dt \\
= & \frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos t+a^4)dt+\frac{1}{2}\int_\pi^{2\pi} \ln(1-2a^2\cos t+a^4)dt \\
= & \frac{1}{2}I(a^2)-\frac{1}{2}\int_\pi^{2\pi} \ln(1-2a^2\cos(2\pi-t)+a^4)d(2\pi-t) \\
= & \frac{1}{2}I(a^2)+\frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos s+a^4)ds \\
= & \frac{1}{2}I(a^2)+\frac{1}{2}I(a^2) \\
= & I(a^2)
\end{aligned}$$

and

$$\begin{aligned}
I(-a) &= \int_0^\pi \ln(1+2a\cos x+a^2)dx \\
&= -\int_\pi^0 \ln(1+2a\cos(\pi-t)+a^2)dt \\
&= \int_0^\pi \ln(1-2a\cos t+a^2)dt \\
&= I(a).
\end{aligned}$$

Hence, inductively, we get

$$I(a)=\frac{1}{2}I(a^2)=\cdots=\frac{1}{2^n}I(a^{2^n}).$$

(d)

$$\begin{aligned} I\left(\frac{1}{a}\right) &= \int_0^\pi \ln\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx \\ &= \int_0^\pi \ln(a^2 - 2a\cos x + 1)dx - \int_0^\pi \ln a^2 dx \\ &= I(a) - 2\pi \ln|a|. \end{aligned}$$

(e) When $|a| < 1$, we have $\lim_{n\to\infty} a^{2^n} = 0$. Then by (b)(i) and (b)(ii),

$$I(a) = \lim_{n\to\infty} \frac{1}{2^n} I(a^{2^n}) = 0.$$

When $|a| > 1$, using the result in (b)(iii), we have

$$I(a) = I\left(\frac{1}{a}\right) + 2\pi \ln|a| = 2\pi \ln|a|.$$

The last equality follows from $I\left(\frac{1}{a}\right) = 0$ since $\frac{1}{|a|} < 1$.