Lecture 2

Recall Lemma 2.2. A natural question is that whether the converse is true, that is if $y \in Y$ is recurrent w.r.t S, is x recurrent w.s.t T for any $x \in \pi^{-1}(y)$? In general it is not true, but it does hold in a special case called group extension.

Definition 2.5 (Group extension)**.** *Let (Y, S) be a TDS and K a compact group. Assume $\psi : Y \to K$ is continuous. Define $X = Y \times K$ and $T : X \to X$ by $T(y, k) = (Sy, \psi(y)k)$. Set $\pi : X \to Y$ by $\pi(y, k) = y$. Then (Y, S) is a factor of (X, T) under factor map π. Say (X, T) is a group extension of (Y, S).*

Examples: 1. For $\alpha \in (0, 1)$, let $(Y, S) = (\mathbb{T}, x \mapsto x + \alpha(\text{mod}1))$. Set (X, T) by $X = \mathbb{T}^2, T(x, y) = (x + \alpha, x + y)$ (From now on, we sometimes omit "$(\bmod 1)$" to ease notation). Then X is a group extension of Y. To see this, let $Y = \mathbb{T}, K = \mathbb{T}$, set $\psi : Y \to K$ by $\psi(y) = y$, then $T(\theta_1, \theta_2) = (\theta_1 + \alpha, \theta_1 + \theta_2) = (\theta_1 + \alpha, \psi(\theta_1) + \theta_2)$.

2. Define $(\mathbb{T}^2, T)$ by $T(\theta_1, \theta_2) = (\theta_1 + \alpha, 2\theta_1 + \theta_2 + \alpha)$. Then $(\mathbb{T}^2, T)$ is a group extension of $(\mathbb{T}, x \mapsto x + \alpha(\text{mod}1))$. To see this, let $Y = K = \mathbb{T}$, define $\psi : Y \to K$ by $\psi(y) = 2y + \alpha$, then $T(\theta_1, \theta_2) = (\theta_1 + \alpha, \psi(\theta_1) + \theta_2)$.

Theorem 2.3. *Let (Y, S) be a TDS and K a compact group. $\psi : Y \to K$ is continuous. Let (X, T) be given by $X = Y \times K, T(y, k) = (Sy, \psi(y)k)$. Then if $y_0 \in Y$ is recurrent w.r.t S, (y_0, k) is recurrent w.r.t T for any $k \in K$.*

Proof. For $k_1 \in K$, define $R_{k_1} : X \to X$ by $R_{k_1}(y, k) = (y, kk_1)$. Note for any $(y, k) \in X$, $R_{k_1}T(y, k) = R_{k_1}(Sy, \psi(y)k) = (Sy, \psi(y)kk_1) = T(y, kk_1) = TR_{k_1}(y, k)$, hence R_{k_1} and T commute. Let e be the identity of K, write $Q(y_0, e) = \overline{\{T^n(y_0, e) : n \geq 1\}}$, then $R_{k_1}(Q(y_0, e)) = Q(R_{k_1}(y_0, e)) = Q(y_0, k_1)$. We first show (y_0, e) is recurrent w.r.t T, it suffices to show $(y_0, e) \in Q(y_0, e)$. Since $T^n(y_0, e) = T^{n-1}(Sy_0, \psi(y_0)) = \cdots = (S^n y_0, \psi(S^{n-1}y_0)\psi(S^{n-2}y_0)\cdots\psi(y_0))$, by assumption that y_0 is recurrent w.r.t S and K is compact, there exist $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}_+$ and $k_1 \in K$ such that $T^{n_i}(y_0, e) \to (y_0, k_1)$, hence $(y_0, k_1) \in Q(y_0, e)$, now act on both sides by R_{k_1}, $R_{k_1}(y_0, k_1) \in R_{k_1}(Q(y_0, e)) = Q(R_{k_1}(y_0, e)) = Q(y_0, k_1)$, that is $(y_0, k_1^2) \in Q(y_0, k_1) \subset Q(y_0, e)$, inductively, $(y_0, k_1), (y_0, k_1^2), \cdots, \in Q(y_0, e)$. We claim (y_0, e) is an accumulation point of $\{(y_0, k_1^n) : n \geq 1\}$. Since K is compact, there exists $\{l_i\}_{i=1}^{\infty} \subset \mathbb{N}_+$, such that $l_i \uparrow \infty$ and $k_1^{l_i} \to b$ for some $b \in K$, then $k_1^{l_{i+1}-l_i} = k_1^{l_{i+1}}(k_1^{l_i})^{-1} \to e$, the claim follows, hence $(y_0, e) \in Q(y_0, e)$, i.e. (y_0, e) is a recurrent point in X. Now for any $k \in K$, we have $(y_0, k) = R_k(y_0, e) \in R_k(Q(y_0, e))) = Q(R_k(y_0, e)) = Q(y_0, k)$, hence (y_0, k) is recurrent in X. □

In the above theorem, the fact that group extension preserves recurrent points finds its applications in number theory, which is first discovered by Furstenberg.

Recall that $(\mathbb{T}^2, T)$ given by $T(\theta_1, \theta_2) = (\theta_1 + \alpha, 2\theta_1 + \alpha + \theta_2)$ is a group extension of $(\mathbb{T}, \theta \mapsto \theta + \alpha(\text{mod}1))$. Since every point in $\mathbb{T}$ is recurrent w.r.t $\theta \mapsto \theta + \alpha(\text{mod}1)$, apply Theorem 2.3, we have

Corollary 2.3.1. *Let $\alpha \in (0,1)$. Any point in $\mathbb{T}^2$ is recurrent w.r.t $T(\theta_1, \theta_2) = (\theta_1 + \alpha, 2\theta_1 + \theta_2 + \alpha)$.*

Now consider the obit of $(0,0)$,

$$(0,0) \to (\alpha, \alpha) \to (2\alpha, 4\alpha) \to \cdots \to (n\alpha, n^2\alpha) \to \cdots,$$

since $(0,0)$ is recurrent, there exists $\{n_i\}_{i=1}^{\infty}$, such that $(n_i\alpha, n_i^2\alpha) \to (0,0)$. Hence we obtain

Corollary 2.3.2. *For any $\alpha \in (0,1)$ and $\epsilon > 0$, there exist $n \geq 1$ and $m \in \mathbb{N}$, such that $|n^2\alpha - m| < \epsilon$.*

More generally, let $P_d(x) \in \mathbb{R}[x]$ be a polynomial of degree d with $P_d(0) = 0$, using the same idea we have

Theorem 2.4. *For any $\epsilon > 0$, there exist $n \geq 1$ and $m \in \mathbb{Z}$ such that*

$$|P_d(n) - m| < \epsilon.$$

Proof. Set $P_{d-1}(x) = P_d(x+1) - P_d(x), P_{d-2}(x) = P_{d-1}(x+1) - P_{d-1}(x), \cdots, P_0(x) = P_1(x+1) - P_1(x)$, then for $k = 0, 1, \cdots, d$, $P_k(x)$ is a polynomial of degree of at most k, in particular $P_0(x) = \alpha$ some constant. For $k = 1, 2, \ldots, d$, define $T_k : \mathbb{T}^k \to \mathbb{T}^k$ by

$$T(\theta_1, \theta_2, \cdots, \theta_k) = (\theta_1 + \alpha, \theta_2 + \theta_1, \cdots, \theta_k + \theta_{k-1}).$$

It's easy to see $(\mathbb{T}^d, T_d)$ is a group extension of $(\mathbb{T}^{d-1}, T_{d-1})$, $\cdots$, $(\mathbb{T}^2, T_2)$ is a group extension of $(\mathbb{T}, \theta \mapsto \theta + \alpha(\text{mod}1))$. Since every point in $\mathbb{T}$ is recurrent w.r.t $\theta \mapsto \theta + \alpha(\text{mod}1)$, by Theorem 2.3, every point in $\mathbb{T}^2$ is recurrent w.r.t T_2, inductively, every point in $\mathbb{T}^d$ is recurrent w.r.t T_d. Note that $T_d(P_1(n), P_2(n), \cdots, P_d(n)) = (P_1(n+1), P_2(n+1), \cdots, P_d(n+1))$, hence

$$T_d^n(P_1(0), P_2(0), \cdots, P_d(0)(=0)) = (P_1(n), P_2(n), \cdots, P_d(n)), \forall n \in \mathbb{N}_+.$$

Since $(P_1(0), P_2(0), \cdots, P_d(0))$ is recurrent w.r.t T_d, there exists $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}_+$, such that $P_d(n_i) \to P_d(0) = 0$, hence for any $\epsilon > 0$, there exists $n \geq 1$, such that $|P_d(n)(\text{mod}1)| < \epsilon$. $\square$

2.4 Minilarity

Definition 2.6 (Minimal TDS). *Let (X, T) be a TDS, say X is minimal if $\overline{\{T^n x : n \geq 1\}} = X$ for all $x \in X$.*

Equivalently, X is said to be minimal if X has no proper non-empty T-invariant compact subset, that is if $Y \subset X$ compact, $Y \neq \emptyset$ and $TY = Y$, then $Y = X$.

Example: For $\alpha \in (0,1)$, define $T_\alpha : x \mapsto x + \alpha(\bmod 1)$ on $\mathbb{T}$. Then if $\alpha \in \mathbb{Q}$, every point in $\mathbb{T}$ is periodic, hence $(\mathbb{T}, T_\alpha)$ is not minimal. If $\alpha \notin \mathbb{Q}$, $(\mathbb{T}, T_\alpha)$ is minimal, this can be seen from the fact that $\{n_i\alpha(\text{mod}1)\}_{i=1}^{\infty}$ is dense in $[0,1]$ for $\alpha \notin \mathbb{Q}$. However, we give a proof as follows.

Proposition 2.2. $(\mathbb{T}, T_\alpha)$ *is minimal if and only if* $\alpha \notin \mathbb{Q}$*.*

Proof. Only " $\Leftarrow$ " needs a proof. Suppose $\alpha \notin \mathbb{Q}$, but T_α is not minimal. Then there exists a non-empty and compact $A \subset \mathbb{T}$, such that $A \neq \mathbb{T}$ and $A + \alpha = A(\text{mod}1)$. Since $\mathbb{T} \setminus A \neq \emptyset$ and is open, we can write $\mathbb{T} \setminus A = \cup_{j=1}^{\infty} I_j$, a countable union of disjoint open intervals. Note $T_\alpha(\mathbb{T} \setminus A) = \mathbb{T} \setminus A$. Assume I is such an interval of the longest length (such I can be found since $\mathbb{T}$ has finite length). There are only three possibilities: (1). $I \cap T_\alpha(I) = \emptyset$; (2). $I \cap T_\alpha(I) \neq \emptyset$ and $T \neq T_\alpha(I)$; (3). $I = T_\alpha(I)$. (2) is impossible, since otherwise $I \cup T_\alpha(I) \subset \mathbb{T} \setminus A$ is an open interval of greater length than I, which contradicts with the choice of I. (3) is also impossible, since otherwise $I = m + T_\alpha(I)$ for some $m \in \mathbb{Z}$, contradicting with $\alpha \notin \mathbb{Q}$. Therefore $I \cap T_\alpha(I) = \emptyset$. Proceed in this way, we get a sequence of pairwise disjoint open intervals $\{I, T_\alpha(I), T_\alpha^2(I), \cdots\}$, with each interval being of the same length, which is impossible since $\mathbb{T}$ is compact. Hence T_α is minimal for $\alpha \notin \mathbb{Q}$.

□

Any topological dynamical system contains a subsystem that is minimal.

Theorem 2.5. *Let* (X, T) *be a TDS, then there exists a compact* $Y \subset X$*,* $Y \neq \emptyset$*, such that* $TY = Y$ *and* $(Y, T|_Y)$ *is minimal.*

Proof. The proof is already contained in Birkhoff Recurrence Theorem. In fact, let $\mathcal{F} = \{Y \subset X \text{ nonemty and compact}, TY \subset Y\}$, partially ordered under set inclusion. As shown in Theorem 2.1, there exists a $Y \in \mathcal{F}$ which is a minimal element of $\mathcal{F}$. Hence $Y \subset X$ compact, $Y \neq \emptyset$ and $TY \subset Y$. Since $T(TY) \subset TY \subset Y$ and TY is compact, $TY \in \mathcal{F}$, whence $TY = Y$ since Y is a minimal element of $\mathcal{F}$. If $A \subset Y$ is non-empty, compact and $TA = A$, then for the same reason, $A = Y$, therefore $(Y, T|_Y)$ is a minimal subsystem. □

2.5 Poincaré Recurrence Theorem

Theorem 2.6 (Poincaré Recurrence Theorem)**.** *Let* $(X, \mathscr{F}, \mu)$ *be a probability space.* $T : X \to X$ *is measurable and preserves* μ*. Let* $B \in \mathscr{F}$ *with* $\mu(B) > 0$*, then almost every point of* B *returns infinitely many often to* B*. That is*

$$\mu(\{x \in B : T^n x \in B \textit{ for infinite many } n \geq 1\}) = \mu(B).$$

Proof. First Note

$$\{x \in B : T^n x \in B \text{ for infinite many } n \geq 1\} = B \bigcap \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} T^{-k} B.$$

For each $n \in \mathbb{N}$, set $B_n = \bigcup_{k=n}^{\infty} T^{-k} B$, then

$$B_0 \supset B_1 \supset B_2 \supset \cdots,$$

since $B \subset B_0$,

$$B = B \cap B_0 \supset B \cap B_1 \supset B \cap B_2 \supset \cdots.$$

Since $T^{-n}B_0 = B_n$ for each $n \in \mathbb{N}$ and T preserves μ,

$$\mu(B_0) = \mu(B_1) = \mu(B_2) = \cdots ,$$

and since μ is a probability measure hence finite, we get

$$\mu(B) = \mu(B \cap B_0) = \mu(B \cap B_1) = \mu(B \cap B_2) \cdots .$$

Therefore

$$\mu\big(B \bigcap \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} T^{-k}B\big) = \mu\big(\bigcap_{n=0}^{\infty}(B \cap B_n)\big) = \lim_{n\to\infty} \mu(B \cap B_n) = \mu(B \cap B_0) = \mu(B),$$

note we have used the fact that μ is finite in the second equality. □