MMAT 5220 Complex Analysis and Its Applications

Lecture 9

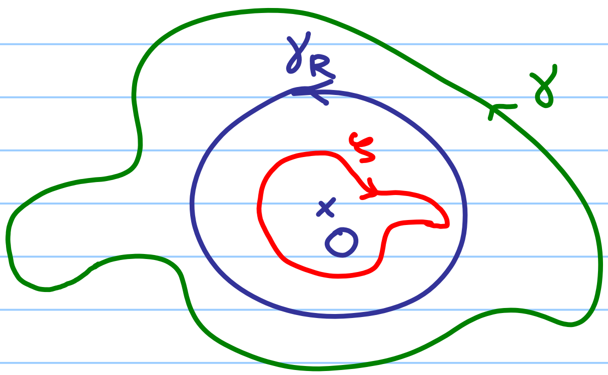
§ Residue (cont'd)

$\boxed{\text{Residues at } \infty}$

Suppose f is analytic in $R < |z| < \infty$ and $\gamma = \gamma(t)$, $a \leq t \leq b$, is a simple closed contour in $R < |z| < \infty$ positively oriented around 0.

Then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$



Now $\varsigma = \varsigma(t) := \dfrac{1}{\gamma(t)}$ is a negatively oriented simple closed contour in $|w| < \frac{1}{R}$. We have

$$\int_\gamma f(z)\,dz = \int_a^b f\left(\frac{1}{\varsigma(t)}\right)\left(-\frac{\varsigma'(t)}{\varsigma(t)^2}\right)dt$$

$$= -\int_a^b g(\varsigma(t))\,\varsigma'(t)\,dt \qquad \text{where} \quad g(w) := \frac{1}{w^2} f\left(\frac{1}{w}\right)$$

$$= \int_{-\varsigma} g(w)\,dw$$

$$= 2\pi i \operatorname*{Res}_{w=0} g(w)$$

(Since $w = \frac{1}{z}$, $|z| > R \Leftrightarrow 0 < |w| < \frac{1}{R}$. So $g(w)$ is analytic for $0 < |w| < \frac{1}{R}$)

This proves :

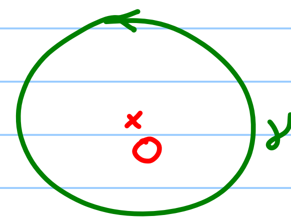
<u>Prop</u> If f is analytic everywhere in $\mathbb{C}$ except at finitely many isolated singular points interior to a positively oriented simple closed contour, then

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname*{Res}_{w=0}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right)\right)$$

Def The residue of f at ∞ is defined as

$$\operatorname*{Res}_{z=\infty} f(z) := -\operatorname*{Res}_{w=0}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right)\right)$$

e.g. $\displaystyle\int_\gamma \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz$ where $\gamma = \{z \in \mathbb{C} : |z| = 3\}$



$f(z) := \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$ has no singular pts outside γ.

$$\frac{1}{w^2} f\left(\frac{1}{w}\right) = \frac{1}{w^2} \frac{\frac{1}{w^3}\left(1-\frac{3}{w}\right)}{\left(1+\frac{1}{w}\right)\left(1+\frac{2}{w^4}\right)} = \frac{w-3}{w(1+w)(2+w^4)} = -\frac{3}{2}\cdot\frac{1}{w} + \cdots$$

$$\Rightarrow \int_\gamma \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i \operatorname*{Res}_{w=0}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right)\right) = -3\pi i$$

Rmk. If $z_1, \dots, z_n$ are the only singular pts of f, then

$$\operatorname*{Res}_{z=\infty} f(z) + \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) = 0$$

(i.e. sum of residues of f over $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is equal to 0.)

$\boxed{\text{Residues at poles}}$

Prop If z_0 is a pole of order $m \geq 1$ of f and we write $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for $0 < |z - z_0| < R$ with $\phi(z)$ analytic and $\phi(z_0) \neq 0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Prop Let p and q be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ & $q'(z_0) \neq 0$.

Then $\displaystyle \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

e.g. For any $n \in \mathbb{Z}$, we have $\cos n\pi = (-1)^n \neq 0$, $\sin n\pi = 0$

and $\displaystyle \frac{d}{dz}(\sin z)\Big|_{z=n\pi} = \cos n\pi \neq 0$.

$$\Rightarrow \operatorname*{Res}_{z=n\pi} \cot z = \operatorname*{Res}_{z=n\pi} \frac{\cos z}{\sin z} = \frac{\cos n\pi}{\cos n\pi} = 1.$$

§ Applications of residues

$\boxed{\text{Computation of improper integrals}}$

Recall : (i) $\displaystyle\int_0^\infty f(x)\,dx := \lim_{R\to\infty} \int_0^R f(x)\,dx$ (if the limit exists)

(ii) $\displaystyle\int_{-\infty}^\infty f(x)\,dx := \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to 0} \int_0^{R_2} f(x)\,dx$

(if both limits exist)

(iii) (Cauchy's Principal Value)

$$\text{P.V.} \int_{-\infty}^\infty f(x)\,dx := \lim_{R\to\infty} \int_{-R}^R f(x)\,dx \quad \text{(if the limit exists)}$$

Rmk $\int_{-\infty}^{\infty} f(x)\,dx$ exists $\underset{\not\Leftarrow}{\Rightarrow}$ P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ exists.

However, if f is an even function, i.e., $f(-x) = f(x)\ \forall x \in \mathbb{R}$,

then $$2\int_0^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx$$

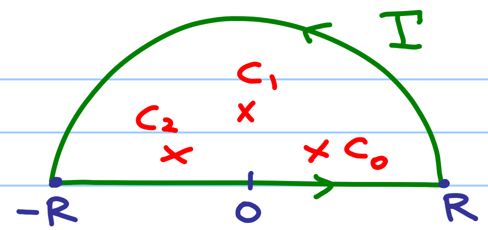
Example 1 Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{x^6+1}$

Sol: Consider $f(z) = \dfrac{1}{z^6+1}$.

Then f is analytic except at the isolated singular points

$$c_k = e^{\frac{(2k+1)\pi i}{6}}, \quad k = 0, 1, \ldots, 5.$$

Let Γ be the contour $l_R + C_R^+$



where l_R is the horizontal line segment from $-R$ to R and C_R^+ is the upper semi-circle of radius R centered at 0.

By Cauchy's Residue Thm,

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=0}^{2} \operatorname*{Res}_{z=c_k} f(z)$$

Note that every c_k is a simple pole of $f(z)$

$$\Rightarrow \operatorname*{Res}_{z=c_k} f(z) = \frac{1}{6c_k^5} = -\frac{c_k}{6}$$

So $$\int_\Gamma \frac{dz}{z^6+1} = 2\pi i\left(-\frac{c_0}{6}-\frac{c_1}{6}-\frac{c_2}{6}\right) = \frac{2\pi}{3}$$

Now $\left| \int_{C_R^+} \frac{dz}{z^6+1} \right| \leq \frac{\pi R}{R^6-1} \to 0$ as $R \to \infty$

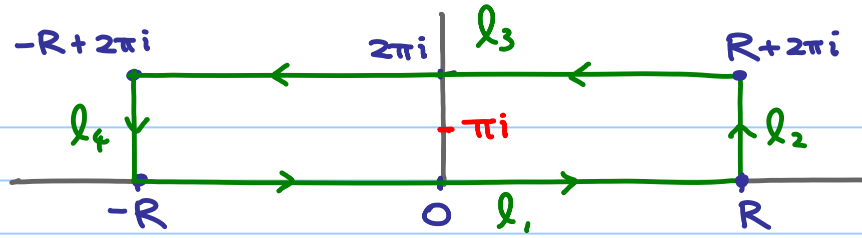
Hence $\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{C_R^+} \frac{dz}{z^6+1} = \frac{2\pi}{3}$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

Since $\frac{1}{x^6+1}$ is even, we have $\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$. #

<u>Example 2</u> Evaluate $P.V. \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ $(0<a<1)$

<u>Sol</u>: Consider $f(z) = \frac{e^{az}}{1+e^z}$ and the following contour $\Gamma = \ell_1 + \ell_2 + \ell_3 + \ell_4$:



Then $z=\pi i$ is the only isolated singular pt interior to Γ.

$$\Rightarrow \int_\Gamma \frac{e^{az}}{1+e^z}\,dz = 2\pi i \operatorname*{Res}_{z=\pi i}\left(\frac{e^{az}}{1+e^z}\right) = -2\pi i e^{a\pi i}$$

Now,

- $$\left|\int_{\ell_2} \frac{e^{az}}{1+e^z}\,dz\right| = \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}}\, i\,dt\right| \leq \frac{2\pi e^{aR}}{e^R-1} \to 0 \text{ as } R\to\infty \quad (\because 0<a<1)$$

- $$\left|\int_{\ell_4} \frac{e^{az}}{1+e^z}\,dz\right| = \left|\int_0^{2\pi} \frac{e^{a(-R+(2\pi-t)i)}}{1+e^{-R+(2\pi-t)i}}(-i)\,dt\right|$$
$$\leq \frac{2\pi e^{-aR}}{1-e^{-R}} \to 0 \text{ as } R\to\infty \quad (\because a>0)$$

- $$\int_{\ell_3} \frac{e^{az}}{1+e^z}\,dz = \int_R^{-R} \frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}}\,dt = -e^{2a\pi i}\int_{-R}^{R} \frac{e^{at}}{1+e^t}\,dt$$

Letting $R \to \infty$, we obtain

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{-2\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \frac{\pi}{\sin a\pi}. \quad \#$$

Improper integrals from Fourier analysis

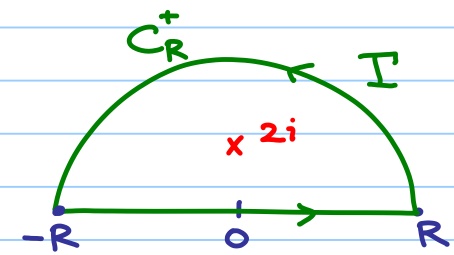
To evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

We may consider contour integrals $\int_{I} f(z)e^{iaz}\,dz$.

Example 3 Evaluate $\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx$ ($f(x)$ decreases fast enough)

Sol : Consider $f(z)\,e^{i2z} = \frac{e^{i2z}}{(z^2+4)^2}$ on Γ



Cauchy integral formula

$$\Rightarrow \int_{-R}^{R} \frac{e^{i2x}}{(x^2+4)^2}\,dx + \int_{C_R^+} \frac{e^{i2z}}{(z^2+4)^2}\,dz = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{i2z}}{(z^2+4)^2}$$

$$= 2\pi i \left.\frac{d}{dz}\left[\frac{e^{i2z}}{(z+2i)^2}\right]\right|_{z=2i} = \frac{5e^{-4}\pi}{16}$$

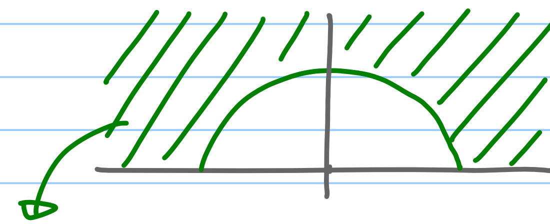
Taking the real parts on both sides

$$\Rightarrow \int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx + \operatorname{Re}\int_{C_R^+} \frac{e^{i2z}}{(z^2+4)^2}\,dz = \frac{5e^{-4}\pi}{16}.$$

Now $\quad \left| \operatorname{Re} \int_{C_R^+} \frac{e^{i2z}}{(z^2+4)^2} dz \right| \leq \left| \int_{C_R^+} \frac{e^{i2z}}{(z^2+4)^2} dz \right| \leq \frac{\pi R}{(R^2-4)^2} \to 0$ as $R \to \infty$.

Since $\frac{\cos 2x}{(x^2+4)^2}$ is even, we have

$$\int_0^\infty \frac{\cos 2x}{(x^2+4)^2} dx = \frac{1}{2} \cdot \frac{5e^{-4}\pi}{16} = \frac{5e^{-4}\pi}{32}. \quad \#$$

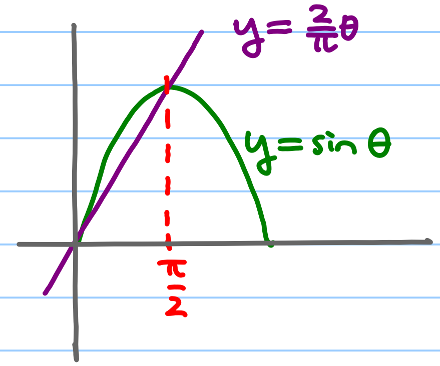


Thm (Jordan's Lemma)

Suppose that f is analytic on $\{x+iy \in \mathbb{C} : y \geq 0 \ \& \ \sqrt{x^2+y^2} \geq R_0\}$ and $\forall R > R_0, \exists M_R \to 0$ as $R \to \infty$ s.t. $|f(z)| \leq M_R$ on $C_R^+ := \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$.

Then $\forall a > 0$, $\displaystyle\lim_{R\to\infty} \int_{C_R^+} f(z) e^{iaz} dz = 0$.

Lemma (Jordan's inequality) $\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$ for $R>0$.



Pf: We have $\frac{2}{\pi}\theta \le \sin\theta$ for $\theta\in[0,\frac{\pi}{2}]$.

$\Rightarrow$ For $R>0$, $e^{-R\sin\theta} \le e^{-\frac{2R}{\pi}\theta}$ $\forall\,\theta\in[0,\frac{\pi}{2}]$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta \le \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\,d\theta = \frac{\pi}{2R}(1-e^{-R}) < \frac{\pi}{2R}.$$

$$\Rightarrow \int_0^{\pi} e^{-2R\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}. \quad \#$$

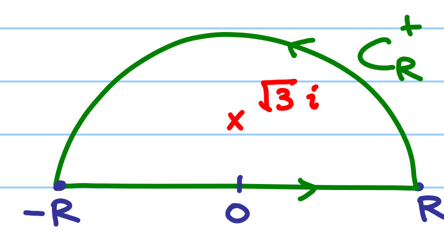
Pf of Thm: Note that $|e^{iaz}| = |e^{iaR(\cos\theta+i\sin\theta)}| = e^{-aR\sin\theta}$

$$\left|\int_{C_R^+} f(z)e^{iaz}\,dz\right| \le M_R\cdot R\cdot\int_0^\pi e^{-aR\sin\theta}\,d\theta < \frac{\pi M_R}{a} \to 0 \text{ as } R\to\infty. \quad \#$$

Example 4 Evaluate $\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx$

Sol: Consider the integral of $f(z)e^{i2z} = \frac{z}{z^2+3}e^{i2z}$ on Γ:

$\sqrt{3}i$ is the only singular pt inside Γ and



$$\operatorname*{Res}_{z=\sqrt{3}i}\left(\frac{ze^{i2z}}{z^2+3}\right) = \frac{1}{2}e^{-2\sqrt{3}}$$

So Cauchy integral formula

$$\Rightarrow \int_{-R}^{R} \frac{xe^{i2x}}{x^2+3}\,dx + \int_{C_R^+} \frac{z}{z^2+3}e^{i2z}\,dz = e^{-2\sqrt{3}}\cdot \pi i$$

On C_R^+, we have $\left|\frac{z}{z^2+3}\right| \le \frac{R}{R^2-3} \longrightarrow 0$ as $R\to\infty$

So Jordan's Lemma $\Rightarrow \int_{C_R^+} \frac{z}{z^2+3}e^{i2z}\,dz \to 0$ as $R\to\infty$

Taking the imaginary parts, we have

$$\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \frac{\pi e^{-2\sqrt{3}}}{2}$$

where we also used the fact that $\frac{x \sin 2x}{x^2+3}$ is even. #