MMAT 5220 Complex Analysis and Its Applications

Lecture 8

§ Zeros and uniqueness of analytic functions (cont'd)

Thm Suppose f is a nonzero analytic function in a domain D and $z_0 \in D$ is a zero of f. Then $\exists\, \varepsilon > 0$ s.t. $f(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$ (meaning that zeros of f are isolated).

Pf: We claim that $\forall\, \varepsilon_1 > 0$, f is not identically zero in $B(z_0, \varepsilon_1) \subset D$.

If so, then Taylor expansion of f at $z_0 \not\equiv 0$

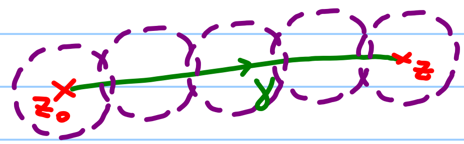
$\Rightarrow$ z_0 is a zero of f of finite order $m \geq 1$.

By the previous thm, we then have $f(z) = (z - z_0)^m g(z)$

where g is analytic and $g(z_0) \neq 0$.

Now continuity of $g \Rightarrow \exists\, \varepsilon > 0$ s.t. $g(z) \neq 0$ in $B(z_0, \varepsilon)$ and hence $f(z) = (z-z_0)^m g(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$.

To prove the claim, suppose $\exists\, \varepsilon_1 > 0$ s.t. $f \equiv 0$ on $B(z_0, \varepsilon_1) \subset D$.
Let $z \in D \setminus \{z_0\}$ and connect z_0 to z by a path γ.
Choose a smaller ε_1, if necessary, so that $\forall\, w \in \gamma$, $B(w, \varepsilon_1) \subset D$.



Now let z_1 be the furthest pt in $\gamma \cap \partial B(z_0, \varepsilon_1)$ along γ.
Then by continuity of f, $f(z_1) = 0$ and it's a non-isolated zero.
By the previous argument, the Taylor expansion of f around z_1, whose radius of convergence $\geq \varepsilon_1$, must be identically zero, so we must have $f \equiv 0$ on $B(z_1, \varepsilon_1)$.

Continuing this process, we see that $f(z)=0$.
Hence $f \equiv 0$ on D, which contradicts our hypothesis. #

Cor Suppose that f and g are analytic in a domain D.
If $f(z)=g(z)$ $\forall z \in E$, where $E \subset D$ contains a limit point which lies in D, then $f(z) \equiv g(z)$ in D.

(Essentially, if $\exists$ a sequence $\{z_n\} \subset D$ s.t. $z_n \longrightarrow z^* \in D$ and $f(z_n)=g(z_n)$ $\forall n$, then $f(z) \equiv g(z)$ in D.)

Pf: The assumptions say that the limit point $z^* \in D$ is a non-isolated zero of $f-g$. Hence the previous thm implies that $f-g \equiv 0$ in D. #

§ Isolated singular points

Def A singular point of f is a point z_0 at which f fails to be analytic. We say that a singular point of f is isolated if $\exists\, \varepsilon > 0$ s.t. f is analytic in $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$.

The Laurent series expansion around z_0 is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Def The series $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is called the principal part of f at z_0.

There are 3 types of isolated singular points :

(1) If $b_n = 0 \ \forall n \geq 1$, then z_0 is called a removable singular point of f.

(2) If $\exists m \geq 1$ s.t. $b_m \neq 0$ and $b_n = 0 \ \forall n \geq m+1$, then z_0 is called a pole of order m of f; a pole of order 1 is called a simple pole

(3) If $b_n \neq 0$ for infinitely many n's, then z_0 is called an essential singular point of f.

e.g. • $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$ has a removable singular point at $z=0$.

• $\frac{1}{z^4(z^2+1)}$ has 3 isolated singular points at $z=0$ (pole of order 4), $z = \pm i$ (simple poles)

- $z=0$ is an essential singular point of $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$.
- $\operatorname{Log} z$ fails to be analytic on $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0 \text{ and } \operatorname{Im} z = 0\}$ but 0 is not an isolated singular point

__Rmk__ Note that if z_0 is a removable singular point of f, then

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots \quad \text{for } 0<|z-z_0|<\varepsilon$$

which can be extended to an analytic function on $|z-z_0|<\varepsilon$ by setting $f(z_0)=a_0$.

__Thm__ Let z_0 be an isolated singular point of f. Then z_0 is a pole of order $m \geq 1$ of f iff $\exists$ analytic g with $g(z_0) \neq 0$ s.t. $f(z) = g(z)/(z-z_0)^m$.

<u>Pf</u> : ($\Rightarrow$) If z_0 is a pole of order m of f, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} \quad \text{for } 0<|z-z_0|<\varepsilon$$

$$\Rightarrow \quad g(z) := (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + b_m$$

has a removable singular point at z_0 and can be extended to an analytic function for $|z-z_0|<\varepsilon$ by the preceding Rmk.

($\Leftarrow$) If $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ and $g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ w/ $a_0 \neq 0$

then by uniqueness of Laurent series, we have

$$f(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots \quad \text{w/ } a_0 \neq 0$$

so f has a pole of order m at z_0. #

e.g. If p and q are analytic and have a zero of order n and m at z_0 respectively, then $f := p/q$ has a

$$\begin{cases} \text{removable singular point} & \text{if } n=m \\ \text{zero of order } n-m & \text{if } n>m \\ \text{pole of order } m-n & \text{if } n<m \end{cases} \quad \text{at } z_0.$$

For instance, $\frac{1-\cos z}{z^2}$ has a removable singular point at z_0.

Cor If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$.

Thm (Riemann) Let z_0 be an isolated singular point of f and suppose f is analytic for $0<|z-z_0|<\varepsilon$. Then z_0 is a removable singular point of f iff f is bounded for $0<|z-z_0|<\varepsilon$.

Pf ($\Rightarrow$) By the maximum modulus principle.

($\Leftarrow$) Write $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

where $b_n = \frac{1}{2\pi i} \int_\gamma f(s)(s-z_0)^{n-1}\, ds$ and $\gamma = \partial B(z_0,\rho)$ for $\rho<\varepsilon$.

f is bounded $\Rightarrow \exists M>0$ s.t. $|f(z)| \leq M \;\; \forall\, 0<|z-z_0|<\varepsilon$.

So $|b_n| \leq \frac{1}{2\pi} M \cdot \rho^{n-1} \cdot 2\pi\rho = M\rho^n \longrightarrow 0$ as $\rho \to 0$ for all $n \geq 1$.

Hence $b_n = 0 \;\; \forall n \geq 1$ and z_0 is a removable singular pt. #

Thm (Casorati-Weierstraß) Suppose z_0 is an essential singular pt of f. Let w_0 be any complex number. Then $\forall\, \varepsilon > 0$ and $\forall\, \delta > 0$, $\exists\, z \in \{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$ s.t. $|f(z) - w_0| < \varepsilon$.

(In other words, for any $\delta > 0$, $f(\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\})$ is dense in $\mathbb{C}$.)

Pf: Suppose not. Then $\exists\, \varepsilon > 0$ and $\delta > 0$ s.t. f is analytic and $|f(z) - w_0| \geq \varepsilon$ for all $0 < |z - z_0| < \delta$.

Then $g(z) := \dfrac{1}{f(z) - w_0}$ is bounded and analytic in $0 < |z - z_0| < \delta$.

$\Rightarrow$ z_0 is a removable singular point of $g(z)$.

If $g(z_0) \neq 0$, then $g(z) \neq 0$ in a nbh of z_0 and hence $f(z) = \dfrac{1}{g(z)} + w_0$ is analytic in a nbh of z_0, which contradicts our assumption.

So we must have $g(z_0)=0$.

Since g is not identically zero, $g(z)=(z-z_0)^m h(z)$ in $|z-z_0|<\delta$ for some $m\geq 1$ and analytic h w/ $h(z_0)\neq 0$.

But then $f(z) = \dfrac{1}{(z-z_0)^m h(z)} + w_0$ in $0<|z-z_0|<\delta$

which has a pole of order m at z_0. This is again a contradiction. #

Rmk • In particular, $\lim_{z\to z_0} f(z)$ doesn't exist

• There is a much stronger version of this thm called the Great Picard Theorem.

§ Residue

Def The residue of f at an isolated singular point z_0 is defined as

$$\operatorname*{Res}_{z=z_0} f(z) := a_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$

i.e. the coefficient of the term $\frac{1}{z-z_0}$ in the Laurent series expansion of f around z_0

e.g. $f(z) = \dfrac{e^z-1}{z^5} = \dfrac{1}{z^4} + \cdots + \dfrac{1}{4!\,z} + \dfrac{1}{5!} + \dfrac{1}{6!}z + \cdots$ for $0<|z|<\infty$

$\Rightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{4!}$

Hence $\int_\gamma f(z)\,dz = \frac{1}{24}$ for any +ve'ly oriented simple closed γ around 0.

e.g. $f(z) = \cosh\left(\frac{1}{z^2}\right) = 1 + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{4!}\cdot\frac{1}{z^8} + \cdots$ for $0 < |z| < \infty$

$\Rightarrow \operatorname*{Res}_{z=0} f(z) = 0$

Hence $\int_\gamma f(z)\,dz = 0$ for any +vely oriented simple closed γ around 0.

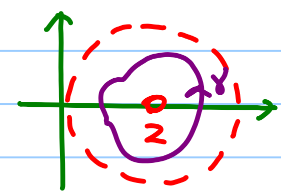
e.g. $f(z) = \frac{1}{z(z-2)^5} = \frac{1}{2(z-2)^5}\left(\frac{1}{1+\frac{z-2}{2}}\right)$

$$= \frac{1}{2(z-2)^5}\left(1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{2(z-2)^5} - \frac{1}{2^2(z-2)^4} + \cdots + \frac{1}{2^5(z-2)} + \cdots$$

$\Rightarrow \operatorname*{Res}_{z=2} f(z) = \frac{1}{32}$

Hence $\int_\gamma f(z)\,dz = \frac{1}{32}$ for any γ like:



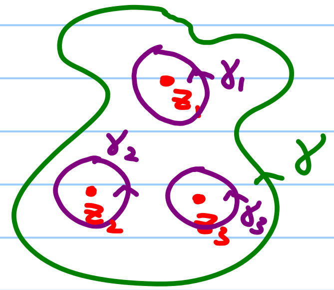
Thm (Cauchy's residue thm)

Let γ be a positively oriented simple closed contour. If f is analytic inside and on γ except for a finite number of singular points $z_1, \ldots, z_n$ inside γ, then

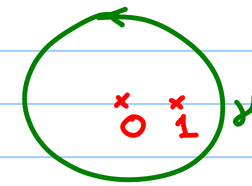
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Pf: For $k=1,\ldots,n$, choose a +ve'ly oriented circle γ_k centered at z_k and small enough so that it's contained inside γ. Then Cauchy-Goursat Thm

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \ \#$$



e.g. $\displaystyle\int_\gamma \frac{4z-5}{z(z-1)}\,dz$ where $\gamma = \{z \in \mathbb{C} : |z| = 2\}$



$$= 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) \right) \quad \text{where} \quad f(z) = \frac{4z-5}{z(z-1)}$$

$$= 2\pi i\,(5-1) = 8\pi i \qquad\qquad = \frac{5}{z} - \frac{1}{z-1}$$

for $0 < |z| < 1$ for $0 < |z-1| < 1$