THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 7 Examples

1. The *Euler numbers* are the numbers E_n $(n = 0, 1, 2, \ldots)$ in the Taylor series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \quad (|z| < \pi/2)$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0 \quad (n = 0, 1, 2, \ldots).$$

Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$

Solution. Notice that

$$\begin{aligned}
\cosh z = 0 \iff & \quad e^z + e^{-z} = 0 \\
\iff & \quad e^z = -e^{-z} \\
\iff & \quad e^{2z} = -1 \\
\iff & \quad 2z = \pi i + 2n\pi i \qquad \text{for some } n \in \mathbb{Z}
\end{aligned}$$

Therefore, the zeros set of $\cosh z$ is $\{\pm\frac{\pi i}{2}, \pm\frac{3\pi i}{2}, \pm\frac{5\pi i}{2}, \ldots\}$. The function $f(z) := \frac{1}{\cosh z}$ is analytic in the disk $\{|z| < \pi/2\}$. Therefore, it admits the Taylor series representation in the disk.

Since $f(-z) = f(z)$, we have $\sum_{n=0}^{\infty} \frac{E_n}{n!}(-1)^n z^n = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$. Uniqueness of Taylor series gives us $-\frac{E_{2n+1}}{(2n+1)!} = \frac{E_{2n+1}}{(2n+1)!}$, hence $E_{2n+1} = 0$ for any $n = 0, 1, 2, \ldots$. Notice that

$$\begin{aligned}
\cosh z = \frac{e^z + e^{-z}}{2} &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
&= 1 - \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \cdots\right)
\end{aligned}$$

Together with the Taylor series expansion

$$\frac{1}{1-w} = 1 + w + w^2 + w^3 + \cdots \quad \text{for } |w| < 1,$$

we have

$$\begin{aligned}
f(z) &= 1 + \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \cdots\right) + \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \cdots\right)^2 + \left(-\frac{z^2}{2!} - \cdots\right)^3 \\
&= 1 + \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \cdots\right) + \left(\frac{z^4}{(2!)^2} + 2\frac{z^6}{2!4!} + \cdots\right) + \left(-\frac{z^6}{(2!)^3} - \cdots\right) \\
&= 1 - \frac{z^2}{2} + \frac{5z^4}{4!} - \frac{61z^6}{6!} + \cdots
\end{aligned}$$

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2. Obtain the Taylor series representation of $\arctan z$ and $\arcsin z$ by consideration of the derived series:

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$
$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2}z^2 + \frac{1\cdot 3}{2\cdot 4}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6 + \cdots,$$

where the branch of the square root function is the principal branch $\{-\pi < \arg z \le \pi\}$.

Solution. Integrating the function $\frac{1}{1+z^2}$ from 0 to w (where $|w| < 1$), we obtain

$$\arctan w = w - \frac{w^3}{3} + \frac{w^5}{5} - \frac{w^7}{7} + \cdots$$

Notice that the power series defines an analytic function $f(w)$ on the disk $\{|w| < 1\}$. Moreover, the function satisfies $f'(w) = \frac{1}{1+w^2}$ for $w \in (-1,1)$ and $f(0) = 0$. Then, we may conclude that $f(w)$ is the usual $\arctan$ function when restricting on $(-1,1)$. In particular, it satisfies

$$\tan(f(w)) = w \quad \text{for } -1 < w < 1.$$

In Week 8, we can see that this implies

$$\tan(f(w)) = w \quad \text{for any } |w| < 1.$$

Similarly, by integrating $\frac{1}{\sqrt{1-z^2}}$ over the line segment between 0 and w (where $|w| < 1$), we obtain

$$\arcsin w = 1 + \frac{1}{2}\frac{w^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{w^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{w^7}{7} + \cdots$$

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3. Develop $\tan z$ in powers of z up to the terms z^7.

Solution. We can use one of those identities $\tan z = \frac{\sin z}{\cos z}$ or $\tan(\arctan w) = w$ to develop the Taylor series of $\tan z$.

For the first method, by Q1, we know that

$$\frac{1}{\cos z} = \frac{1}{\cosh(iz)} = 1 - \frac{(iz)^2}{2} + \frac{5(iz)^4}{4!} - \frac{61(iz)^6}{6!} + \cdots$$
$$= 1 + \frac{z^2}{2} + \frac{5z^4}{4!} + \frac{61z^6}{6!} + \cdots$$

Therefore,

$$\begin{aligned}
\tan z &= \sin z\left(\frac{1}{\cos z}\right) \\
&= (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots)(1 + \frac{z^2}{2} + \frac{5z^4}{4!} + \frac{61z^6}{6!} + \cdots) \\
&= z + z^3(\frac{1}{2} - \frac{1}{3!}) + z^5(\frac{5}{4!} - \frac{1}{3!2} + \frac{1}{5!}) + z^7(\frac{61}{6!} - \frac{5}{3!4!} + \frac{1}{5!2} - \frac{1}{7!}) + \cdots \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots
\end{aligned}$$

We can also employ the Taylor series of $\arctan w$ to find the Taylor series representation of $\tan z$. Since $\tan$ is an odd function, we may suppose

$$\tan z = a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots$$

Note that by Q2,

$$\begin{aligned}\tan(\arctan w) &= a_1(w - \frac{w^3}{3} + \frac{w^5}{5} - \frac{w^7}{7} + \dots) + a_3(w - \frac{w^3}{3} + \frac{w^5}{5} + \cdots)^3 \\ &+ a_5(w - \frac{w^3}{3} + \dots)^5 + a_7(w + \dots)^7\end{aligned}$$

Up to the terms w^7, we have

$$\begin{aligned}(w - \frac{w^3}{3} + \frac{w^5}{5} + \cdots)^3 &= w^3 + 3w^2(-\frac{w^3}{3})^1 + 3w^2(\frac{w^5}{5})^1 + 3w(-\frac{w^3}{3})^2 + \cdots \\ &= w^3 - w^5 + \frac{14}{15}w^7 + \cdots\end{aligned}$$

and

$$\begin{aligned}(w - \frac{w^3}{3} + \frac{w^5}{5} + \cdots)^5 &= w^5 + 5w^4(-\frac{w^3}{3}) + \cdots \\ &= w^5 - \frac{5}{3}w^7 + \cdots\end{aligned}$$

In order to have $\tan(\arctan w) = w$, we have

$$\begin{aligned}a_1 &= 1 \\ -\frac{a_1}{3} + a_3 &= 0 \\ \frac{a_1}{5} - a_3 + a_5 &= 0 \\ -\frac{a_1}{7} + \frac{14a_3}{15} - \frac{5a_5}{3} + a_7 &= 0\end{aligned}$$

That is, $a_1 = 1$, $a_3 = 1/3$, $a_5 = 2/15$, $a_7 = 17/315$. ◀

4. Suppose f is an entire function and such that for each $z_0 \in \mathbb{C}$, at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

Solution. By the assumption and the hint, for every $z_0 \in \mathbb{C}$, we can find some $n \in \mathbb{N} \cup \{0\}$ such that $f^{(n)}(z_0) = 0$. Therefore, we have

$$\bigcup_{n=0}^{\infty} \{z \in \mathbb{C} : f^{(n)}(z) = 0\} = \mathbb{C}.$$

Since $\mathbb{C}$ is uncountable, at least one set in LHS must be uncountable, say the set $\{z \in \mathbb{C} : f^{(n_0)}(z) = 0\}$ is uncountable.

By the same argument and consider $\bigcup_{n=1}^{\infty} \{z : |z| \leq n, f^{(n_0)}(z) = 0\}$, for some $N \in \mathbb{N}$, the set $\{z : |z| \leq N, f^{(n_0)}(z) = 0\}$ is uncountable.

The analytic function $f^{(n_0)}$ having infinitely many zeros in the closed ball $\{z : |z| \leq N\}$, must be a zero function. This shows that f is a polynomial with degree less than n_0.

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