THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 4
Due Date: 9th April 2020

Compulsory Part

1. Expand e^z into a Taylor series about the point $z = 1$.

   Solution. Note that

$$e^z = e\, e^{z-1} = \sum_{n=0}^{\infty} \frac{e(z-1)^n}{n!}.$$

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2. Show that the Laurent series of $\dfrac{e^z}{z(z^2+1)}$ is given by

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots$$

   for $0 < |z| < 1$.

   Solution. Notice that

$$\begin{aligned}
\frac{e^z}{z} &= \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{3!}z^2 + \cdots \\
\frac{1}{1+z^2} &= \frac{1}{1-(-z^2)} = 1 - z^2 + (-z^2)^2 + (-z^2)^3 + \cdots \\
&= 1 - z^2 + z^4 + \cdots
\end{aligned}$$

   Therefore, we have

$$\begin{aligned}
\frac{e^z}{z(z^2+1)} &= \frac{1}{z}(1 - z^2 + z^4 + \cdots) + (1 - z^2 + \cdots) + \frac{1}{2}z(1 - z^2 + \cdots) \\
&\quad + \frac{1}{3!}z^2(1 - z^2 + \cdots) + \cdots \\
&= \frac{1}{z} + 1 + z(-1 + \frac{1}{2}) + z^2(-1 + \frac{1}{3!}) + \cdots \\
&= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots
\end{aligned}$$

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3. Find the Laurent series of $\dfrac{1}{(z-1)(z-2)}$ in

   (a) $|z| < 1$;

   (b) $1 < |z| < 2$;

(c) $1 < |z-3| < 2$.

Solution. By partial fraction, we have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

(a) In the domain $|z| < 1$,

$$\begin{aligned}\frac{1}{z-1} &= -1 - z - z^2 - z^3 + \cdots = -\sum_{k=0}^{\infty} z^k \\ \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)} \\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}\end{aligned}$$

Hence, we have

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty}\left(1 - \frac{1}{2^{k+1}}\right) z^k.$$

(b) In the domain $1 < |z| < 2$,

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} \\ \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}\end{aligned}$$

Hence, we have

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty}\frac{-z^k}{2^{k+1}} - \sum_{k=1}^{\infty}\frac{1}{z^k}$$

(c) In the domain $1 < |z-3| < 2$,

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{(z-3)+2} = \frac{1}{2}\frac{1}{1-\left(\frac{3-z}{2}\right)} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(3-z)^k}{2^k} \\ \frac{1}{z-2} &= \frac{1}{(z-3)+1} = \frac{1}{z-3}\frac{1}{1-\left(\frac{1}{3-z}\right)} \\ &= \frac{1}{z-3}\sum_{k=0}^{\infty}\frac{1}{(3-z)^k}\end{aligned}$$

Hence, we have

$$\frac{1}{(z-1)(z-2)} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(z-3)^k} - \sum_{k=0}^{\infty}\frac{(-1)^k(z-3)^k}{2^{k+1}}$$

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4. Show that the function $f(z) = 1 - \cos z$ has a zero of order 2 at $z_0 = 0$.

   Solution. $f'(z) = \sin z$ and $f''(z) = \cos z$. Since $f(0) = f'(0) = 0$ and $f''(0) = 1 \neq 0$, $f(z)$ has a zero of order 2 at $z_0 = 0$. ◀

5. Suppose that $f(z)$ and $g(z)$ are functions analytic at z_0. If z_0 is a zero of both $f(z)$ and $g(z)$ of order $m \geq 1$, show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}.$$

   Solution. Since $f(z), g(z)$ has a zero of order $m \geq 1$ at z_0, for z near z_0, we have the Taylor series expansion

$$f(z) = \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \cdots$$
$$g(z) = \frac{g^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{g^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \cdots$$

   where $f^{(m)}(z_0), g^{(m)}(z_0) \neq 0$. Therefore,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f^{(m)}(z_0) + \frac{f^{(m+1)}(z_0)}{m+1}(z - z_0) + \cdots}{g^{(m)}(z_0) + \frac{g^{(m+1)}(z_0)}{m+1}(z - z_0) + \cdots} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$$

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Optional Part

1. With the aid of series, prove that the function f defined by

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

   is an entire function.

   Solution. Note that by the Taylor series expansion,

$$\frac{e^z - 1}{z} = \frac{1}{z}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} - 1\right) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} \quad \text{for } z \neq 0.$$

   The power series above defines an entire function attaining 1 at $z = 0$. This shows that the function $f(z)$ defined in the question is an entire function, which is precisely the power series. ◀

2. Let f be a function analytic in a domain $D \subset \mathbb{C}$ which has distinct zeros $z_1, z_2, \ldots, z_n$ of orders $m_1, m_2 \ldots, m_n$ respectively. Show that there exists an analytic function $g(z)$ on D such that
$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_n)^{m_n} g(z).$$

Solution. Both functions $f(z)$ and $(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}$ have zeros $z_1, z_2, \ldots, z_n$ of orders $m_1, m_2 \ldots, m_n$ respectively. Hence the function $\frac{f(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}}$ has removable singularities at $z_1, z_2, \ldots, z_n$ (see Week 8 Lecture). Therefore, there are analytic functions $g_1, g_2, \cdots, g_n$ around $z_1, z_2, \ldots, z_n$ such that

$$g_i(z) = \frac{f(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}} \quad \text{for } 0 < |z - z_i| < \epsilon$$

If we put

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}} & \text{if } z \neq z_1, z_2 \ldots, z_n, \\ g_i(z_i) & \text{if } z = z_i \text{ for } i = 1, 2, \ldots n. \end{cases}$$

then $g(z)$ is the desired function.

We can also do it by induction. The arguments are essentially the same.

Since $f(z)$ has a zero of order m_1 at z_1, by Week 7 Lecture, we can find a small disk around z_1, and an analytic function $G_1(z)$ such that $f(z) = (z-z_1)^{m_1} G_1(z)$ on the small disk, moreover, $G_1(z_1) \neq 0$. The formula shows that the function $g_1(z)$ defined by

$$g_1(z) = \begin{cases} \dfrac{f(z)}{(z-z_1)^{m_1}} & \text{if } z \neq z_1, \\ G_1(z_1) & \text{if } z = z_1. \end{cases}$$

is analytic on D. Indeed, $g_1(z) = G_1(z)$ around z_1, hence is analytic at z_1. For $z \neq z_1$, g_1 is analytic because f is analytic. Therefore, there exists an analytic function $g_1(z)$ on D such that $f(z) = (z - z_1)^{m_1} g_1(z)$. Applying the same argument to $g_1(z)$, there is an analytic function $g_2(z)$ on D such that $g_1(z) = (z - z_2)^{m_2} g_2(z)$. Inductively, there exists an analytic function $g(z)$ on D such that

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_n)^{m_n} g(z).$$

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3. Let R be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ at z_0. Show, by term-by-term differentization and mathematical induction, that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} (z - z_0)^n$$

for $|z - z_0| < R$.

Solution. For $n = 1$, by termwise differentiation,

$$f'(z) = \sum_{n=1}^{\infty} a_n n(z - z_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n + 1)(z - z_0)^n = \sum_{n=0}^{\infty} \frac{(1 + n)!}{n!} a_{1+n}(z - z_0)^n$$

Assume it is true for $m = k$, i.e.

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k + n)!}{n!} a_{k+n}(z - z_0)^n.$$

Then, we have

$$f^{(k+1)}(z) = \sum_{n=1}^{\infty} \frac{(k + n)!}{n!} a_{k+n} n(z - z_0)^{n-1} = \sum_{n=0}^{\infty} \frac{(k + 1 + n)!}{n!} a_{k+1+n}(z - z_0)^n.$$

By induction, the statement is true for every $m \in \mathbb{N}$. ◀

4. Let f be an entire function such that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for all $x \in \mathbb{R}$. Show that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

for all $z \in \mathbb{C}$.

Solution. Recall that if $\sum_{n=0}^{\infty} c_n z^n$ is a power series converging for some $z = z_0$, then it is absolutely convergent for every $|z| < |z_0|$.

Since $\sum_{n=0}^{\infty} a_n z^n$ converges for every $z \in \mathbb{R}$, it converges absolutely for every $z \in \mathbb{C}$.

Moreover, if we put $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $g(z)$ is an entire function coinciding with $f(z)$ on the real axis. Hence, $f - g$ is an entire function with non-isolated zeros. Therefore, we can conclude that $f - g \equiv 0$ on $\mathbb{C}$. ◀