THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 3
Due Date: 19th March 2020

Compulsory Part

1. Let γ be a positively oriented circle which does not pass through $z_0 \in \mathbb{C}$. Show that

$$\int_\gamma \frac{dz}{z - z_0} = \begin{cases} 2\pi i & \text{if } z_0 \text{ lies inside } \gamma, \\ 0 & \text{if } z_0 \text{ lies outside } \gamma. \end{cases}$$

Solution. If z_0 lies outside γ, then the function $1/(z - z_0)$ is analytic at all points interior to and on the contour γ. By Cauchy-Goursat theorem, we have $\int_\gamma \frac{dz}{z-z_0} = 0$. If z_0 lies inside γ, we can apply the Cauchy integral formula to the constant function 1, which gives $\int_\gamma \frac{dz}{z-z_0} = 2\pi i$. ◀

2. Let γ be the positively oriented (i.e. going in the counterclockwise direction) boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of the following integrals:

(a) $\displaystyle\int_\gamma \frac{e^{-z}}{z - (\pi i/2)} dz$

(b) $\displaystyle\int_\gamma \frac{\cos z}{z(z^2 + 8)} dz$

Solution.

(a) Note that $|\pi/2| < 2$, hence $\pi i/2$ lies inside γ. Also, the function e^{-z} is analytic at all points interior to and on the contour γ. Applying the Cauchy integral formula to the function e^{-z}, we have

$$\int_\gamma \frac{e^{-z}}{z - (\pi i/2)} dz = 2\pi i(e^{-\pi i/2}) = 2\pi i(-i) = 2\pi.$$

(b) Note that $z^2 + 8 = (z - 2\sqrt{2}i)(z + 2\sqrt{2}i)$. Since $|2\sqrt{2}| > 2$, both $\pm 2\sqrt{2}i$ lie outside the contour γ. Therefore, $\frac{\cos z}{z^2+8}$ is an analytic function at all points interior to and on the contour γ. Applying the Cauchy integral formula, we have

$$\int_\gamma \frac{\cos z}{z(z^2 + 8)} dz = 2\pi i \left(\frac{\cos 0}{0^2 + 8} \right) = \pi i/4$$

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3. Let $a \in \mathbb{R}$. By integrating the function e^{az}/z around the unit circle, parametrized as $\gamma(\theta) = e^{i\theta}$, $-\pi \le \theta \le \pi$, show that

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi.$$

Solution. Using the parametrization $\gamma(\theta) = e^{i\theta}$, $-\pi \le \theta \le \pi$, we have

$$\begin{aligned}\int_\gamma \frac{e^{az}}{z} dz &= \int_{-\pi}^{\pi} \frac{e^{a(e^{i\theta})}}{e^{i\theta}} ie^{i\theta} d\theta \\ &= i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)} d\theta \\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + i\sin(a\sin\theta))d\theta \\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta \\ &= 2i\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta\end{aligned}$$

Last equality is due to the fact that $e^{a\cos\theta}\cos(a\sin\theta)$ is an even function while $e^{a\cos\theta}\sin(a\sin\theta)$ is an odd function.

On the other hand, since e^{az} is entire, the Cauchy integral formula yields

$$\int_\gamma \frac{e^{az}}{z} dz = 2\pi i(e^{a(0)}) = 2\pi i$$

The result follows by equating the two equations. ◀

4. Let $n \in \mathbb{Z}$ and γ be the positively oriented unit circle. Compute $\displaystyle\int_\gamma \frac{e^z}{z^n} dz$. (Hint: there are two cases to be considered.)

Solution. For $n \le 0$, e^z/z^n is an entire function. Cauchy-Goursat theorem tells us that $\int_\gamma \frac{e^z}{z^n} dz = 0$. For $n \ge 1$, we will employ the Cauchy integral formula:

$$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{m+1}} dz \quad \text{for any } m \ge 0,$$

to the function $f(z) = e^z$ with $z_0 = 0$. This gives $\int_\gamma \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = \frac{2\pi i}{(n-1)!}$. ◀

5. Let $f(z)$ be an entire function.
   (a) If $f^{(n)}(z) \equiv 0$ for some $n \in \mathbb{N}$, show that $f(z)$ is a polynomial.
   (b) Prove that if $|f(z)| < |z|^n$ for all $|z| > R$, where $R > 0$ and $n \in \mathbb{N}$, then $f(z)$ must be a polynomial. (Hint: Use the Cauchy integral formula to estimate $f^{(n+1)}(z)$.)

Solution.

(a) Since $f(z)$ is an entire function, we have the Taylor series representation

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

in the whole complex plane, where $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-0)^{k+1}} ds = \frac{f^{(k)}(0)}{k!}$. The contour γ is a positively oriented simple closed contour and its interior contains 0.
Since $f^{(n)}(z) \equiv 0$, we have $f^{(k)}(z) \equiv 0$ for all $k \ge n$. In particular, $f^{(k)}(0) = 0$. Therefore, $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$ is a polynomial.

(b) We will show that $f^{(n+1)}(z) \equiv 0$ using the Cauchy integral formula. First fix some $z_0 \in \mathbb{C}$, and consider any $M > 0$ such that $M > \max(|z_0|, R)$. The element z_0 is inside the positively oriented contour $\gamma_M := \{z \in \mathbb{C} : |z| = M\}$. By Cauchy integral formula, we have

$$\begin{aligned} \left|f^{(n+1)}(z_0)\right| &= \left|\frac{(n+1)!}{2\pi i}\int_{\gamma_M}\frac{f(z)}{(z-z_0)^{n+2}}dz\right| \\ &\leq \frac{(n+1)!}{2\pi}\int_{\gamma_M}\frac{|f(z)|}{|z-z_0|^{n+2}}dz \\ &\leq \frac{(n+1)!}{2\pi}\int_{\gamma_M}\frac{|z|^n}{(|z|-|z_0|)^{n+2}}dz \\ &= \frac{(n+1)!}{2\pi}\frac{M^n}{(M-|z_0|)^{n+2}}2\pi M \end{aligned}$$

Notice that both n and z_0 is fixed. Letting $M \to \infty$, we would obtain $f^{(n+1)}(z_0) = 0$, and this holds for any $z_0 \in \mathbb{C}$. Therefore, $f^{(n+1)}(z) \equiv 0$. By part (a), we conclude that f must be a polynomial with degree less than n.

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6. Suppose that $f(z)$ is entire and there exists $A > 0$ such that $|f(z)| \leq A|z|$ for all $z \in \mathbb{C}$. Show that $f(z) = az$ for some constant $a \in \mathbb{C}$.

Solution. In solution of Q5(a), we have shown that if f is an entire function and $f^{(n)}(z) \equiv 0$, then f is a polynomial with degree less than n. In this question, we can use the Cauchy integral formula and the given inequality to claim that $f^{(2)}(z) \equiv 0$ as in Q5(b). Hence, $f(z) = a_0 + a_1 z$. The given inequality also suggests that $f(0) = a_0 = 0$. This gives the desired result.

Here is another way to argue. Since $f(z)$ is an entire function, it admits the Taylor series representation $a_0 + a_1 z + a_2 z^2 + \cdots$ on the whole complex plane, where $f(0) = a_0 = 0$ by the given inequality. Therefore, the Taylor series after divided by z is still a power series, i.e. the coefficients satisfy $a_{-n} = 0$ for $n \geq 1$. It is an entire function coinciding with $\frac{f(z)}{z}$ for $z \neq 0$. We may call it $f_1(z)$. By the given inequality, $|f_1(z)| \leq A$ for any $z \neq 0$. Liouville's Theorem (every bounded entire function is a constant function) shows that $f_1(z) \equiv a$ for some complex number a. In conclusion, we have $f(z) = az$ for all $z \neq 0$. ◀

Optional Part

1. Let γ be a simple closed contour in $\mathbb{C}$, $R \subset \mathbb{C}$ be the interior of γ, and f be a continuous function on γ. Show that the function

$$F(z) := \int_\gamma \frac{f(s)}{s-z}ds,$$

defined for $z \in R$, is analytic on R with

$$F'(z) = \int_\gamma \frac{f(s)}{(s-z)^2}ds$$

for $z \in R$.

Solution. From now on, we fix $z \in R$, and notice that

$$F(z+h) - F(z) = \int_\gamma \frac{f(s)}{s-(z+h)} - \frac{f(s)}{s-z} ds$$
$$= \int_\gamma f(s) \frac{h}{(s-z-h)(s-z)} ds$$

and then

$$\frac{F(z+h)-F(z)}{h} - \int_\gamma \frac{f(s)}{(s-z)^2} ds = \int_\gamma f(s) \left(\frac{1}{(s-z-h)(s-z)} - \frac{1}{(s-z)^2} \right) ds$$
$$= \int_\gamma f(s) \left(\frac{h}{(s-z)^2(s-z-h)} \right) ds$$

(1) Since f is continuous on γ, there is $M > 0$ such that $|f(s)| \leq M$ for all $s \in \gamma$.

(2) z is some point interior to γ, so there is $\delta > 0$ so that $|s-z| \geq \delta$ for all $s \in \gamma$. i.e. z is kept away from the contour γ.

(3) For $|h| < \delta/2$, we have $|s-z-h| \geq |s-z| - |h| \geq \delta/2$.

Therefore,

$$\left| \frac{F(z+h)-F(z)}{h} - \int_\gamma \frac{f(s)}{(s-z)^2} ds \right| \leq \int_\gamma |f(s)| \frac{|h|}{|s-z|^2|s-z-h|} ds$$
$$\leq M \frac{|h|}{\delta^2(\delta/2)} \cdot \text{length of } \gamma$$

Letting $h \to 0$, we have RHS$\to 0$. That is,

$$F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \int_\gamma \frac{f(s)}{(s-z)^2} ds$$

for $z \in R$. ◀

2. By integrating the function

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$$

around the unit circle, parametrized as $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, show that for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} t \, dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

Solution. Using the parametrization $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, we have

$$\int_\gamma \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} dz = \int_0^{2\pi} \frac{1}{e^{it}} \left(e^{it} + \frac{1}{e^{it}}\right)^{2n} e^{it} i \, dt$$
$$= i \int_0^{2\pi} (e^{it} + e^{-it})^{2n} \, dt$$
$$= i2^{2n} \int_0^{2\pi} \cos^{2n} t \, dt$$

On the other hand,

$$\begin{aligned}\frac{1}{z}\left(z+\frac{1}{z}\right) &= \frac{1}{z}\left(\sum_{k=0}^{2n}\binom{2n}{k}z^k\left(\frac{1}{z}\right)^{2n-k}\right)\\ &= \sum_{k=0}^{2n}\binom{2n}{k}z^{2(k-n)-1}\end{aligned}$$

Recall that (this can be calculated directly, or you may argue that for any $n \neq -1$, z^n has an antiderivative in $\mathbb{C} \setminus \{0\}$.)

$$\int_\gamma z^n dz = \begin{cases} 2\pi i & \text{if } n = -1; \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\int_\gamma \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz = 2\pi i\binom{2n}{n} = 2\pi i\frac{1\cdot 2\cdots 2n}{(1\cdot 2\cdots n)^2}$$

Equating the two equations, we obtain

$$\begin{aligned}\int_0^{2\pi}\cos^{2n} t\, dt &= \frac{2\pi}{2^{2n}}\frac{1\cdot 2\cdots 2n}{(1\cdot 2\cdots n)^2}\\ &= 2\pi\frac{1\cdot 2\cdots 2n}{(2\cdot 4\cdots (2n))^2}\\ &= 2\pi\frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots (2n)}.\end{aligned}$$

The result follows by dividing 2π on both sides. ◀

3. Suppose that $f(z)$ is entire and there exists $M \in \mathbb{R}$ such that Re$f(z) \leq M$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a constant function.

Solution. By chain rule, we see that the composite function

$$e^{f(z)} = e^{\mathrm{Re}\, f(z)}(\cos(\mathrm{Im}\, f(z)) + i\sin(\mathrm{Im}\, f(z)))$$

is an entire function. Moreover, $|e^{f(z)}| = e^{\mathrm{Re}\, f(z)} \leq e^M$ for every z. Due to Liouville's theorem, $e^{f(z)}$ is a constant function, say $e^{f(z)} = C$, where C is a nonzero complex number. For each $z \in \mathbb{C}$, we have

$$f(z) = \log C + 2\pi i n \quad \text{for some } n \in \mathbb{Z}.$$

Notice that at this stage, different z may correspond to different $n \in \mathbb{Z}$. We need to argue that all z share the same $n \in \mathbb{Z}$ by the continuity of f. Loosely speaking, continuity of f guarantees that the function $f(z)$ cannot jump from $\log C + 2\pi i n_1$ to $\log C + 2\pi i n_2$ without taking on any other values in between. This shows that $f(z)$ is a constant function. ◀

4. Suppose that f is analytic in $|z| \leq R$ and there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $|z| \leq R$. Show that, for any $n \in \mathbb{N}$, we have

$$\left|f^{(n)}(z)\right| \leq \frac{n!MR}{(R-|z|)^{n+1}}$$

for all $|z| < R$.

Solution. Consider the postively oriented contour $\gamma = \{z \in \mathbb{C} : |z| = R\}$. Since f is analytic at all points interior to and on the contour, by the Cauchy integral formula, we have

$$\begin{aligned} |f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^{n+1}}\,ds\right| \qquad \text{for any } |z| < R, \\ &\leq \frac{n!}{2\pi}\int_\gamma \frac{|f(s)|}{(|s|-|z|)^{n+1}}\,ds \\ &\leq \frac{n!}{2\pi}\frac{M}{(R-|z|)^{n+1}}(2\pi R) = \frac{n!MR}{(R-|z|)^{n+1}}. \end{aligned}$$

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