Recall:

Introduce Jacobi field $J(t)$ along a given geodesic $\gamma : [a,b] \to M$, which is given by

$$\nabla_{\gamma'}\nabla_{\gamma'} J + R(J,\gamma')\gamma' = 0 \quad \text{on } [a,b].$$

- $J(t)$ is uniquely determined by $J(0)$, $J'(0)$

$\Downarrow$

- If $\gamma(t) = \exp_p(tv)$ for some $v \in T_pM$.

and $J(0) = 0$, $J'(0) = w \in T_pM$. then

$$J(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(t v(s)) \quad \text{where} \begin{cases} v(0) = v \\ v'(0) = w. \end{cases}$$

---

prop (asymptotics of J) If $J(0) = 0$ and $J'(0) = w$, $|w| = 1$, then

$$|J(t)|^2 = t^2 - \underbrace{\tfrac{1}{3} R(w,v,v,w)\, t^4}_{\text{Next leading term}} + o(t^4), \text{ as } t \to 0.$$

pf: $J(0) = 0$, $|J|^2 = f(t)$, $\Rightarrow \begin{cases} f(0) = 0 \\ f'(0) = 0 \end{cases}$

$$f'' = (2\langle J', J\rangle)' = 2\langle \overset{\nabla_{\gamma'} J}{J'}, J'\rangle + 2\langle \overset{\nabla_{\gamma'}\nabla_{\gamma'} J}{J''}, J\rangle$$

① at $t=0$, $f''(0) = 2\langle J'(0), J'(0)\rangle = 2|w|^2 = 2$.

② $$f'''(0) = 2\langle \overset{\text{2nd}}{J''}, J'\rangle + 2\langle J', \overset{\text{2nd}}{J''}\rangle + 2\langle J''', \overset{0}{J}\rangle + 2\langle \overset{\text{2nd}}{J''}, J'\rangle \equiv 0$$

Since $J'' = -R(\overset{0}{\cancel{J}}, \gamma')\gamma' = 0$ at $t = 0$

③ $$f''''(0) = 2\langle J''', J'\rangle + 2\langle \cancel{J'', J''}\rangle + 2\langle \cancel{J'', J''}\rangle + 2\langle J', J'''\rangle + 2\langle \cancel{J'''', J}\rangle + 2\langle J''', J'\rangle + 2\langle J''', J'\rangle + 2\langle \cancel{J'', J''}\rangle$$

$$= 8\langle J''', J'\rangle$$

$$= -8\langle (R(J,\gamma')\gamma')', J'\rangle$$

$$= -8\langle (\nabla_{\gamma'}R)(\cancelto{0}{J}, \gamma')\gamma' + R(J',\gamma')\gamma' + R(J', \cancelto{0}{\nabla_{\gamma'}\gamma'})\gamma' + R(J',\gamma')(\cancelto{0}{\nabla_{\gamma'}\gamma'}), J'\rangle$$

$$= -8R(J',\gamma',\gamma',J')$$

$$= -8R(w,v,v,w)$$ #

S^n $\boxed{\text{Great circle = geodesic!!}}$

☆☆ exp map ≠ diff

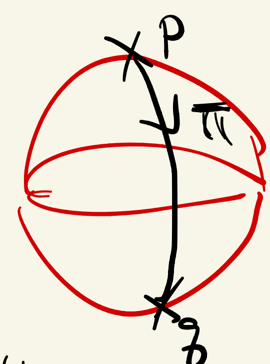
"After passing through q".

Defn: Along a geodesic $\gamma : [0, l] \to M$

① Along γ, $t_0 \in [0, l]$. We say that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ if $\exists$ Jacobi field $J \not\equiv 0$ s.t. $J(0) = 0 = J(t_0)$.

② Max No. of linearly indep. those J $=$ multiplicity of conjugate pt $\gamma(t_0)$.

Example: S^n, the Jacobi-field can be solved by ODE. where



$$J(t) = \sin(t) \cdot w(t) \quad (J(0) = 0)$$

where $|w(t)| = 1$, $w(t) \perp \gamma'(t)$

$\Rightarrow$ q is conjugate to p, multiplicity $= n-1$.

Prop: $\gamma(t_0) =$ conjugate to $\gamma(0)$ iff

$t_0 \gamma'(0) =$ critical value of $\exp_p$.

pf: If $\gamma(t_0)$ = conjugate pt, then $\exists$ JF.

J along $\gamma|_{[0,t_0]}$ s.t. $J(0) = J(t_0) = 0$

let $J'(0) = w$, then uniqueness lemma implies

that $0 \neq J(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(tv(s)) \triangleq d\exp_p\big|_{t_0 v}(t_0 w)$

$J(t_0) = 0 \Leftrightarrow d\exp_p\big|_{t_0 v}(t_0 \underline{w}) = 0$ for some w

$(w \neq 0) \Leftrightarrow t_0 v = t_0 \gamma'(0) =$ critical value $\#$

---

prop: $\gamma : [0,l] \to M$ is a geodesic s.t.

$\gamma(l)$ is Not a conjugate pt of $\gamma(0)$.

If $v \in T_{\gamma(0)}M$, $w \in T_{\gamma(l)}M$, then $\exists!$ JF.

$J(t)$ along $\gamma(t)$ s.t. $J(0) = v$, $J(l) = w$.

pf: $\boxed{\text{Uniqueness}}$: If $\exists$ JF, $J(t)$ and $\tilde{J}(t)$

s.t. $J(0) = \tilde{J}(0) = v$, $J(l) = \tilde{J}(l) = w$

then $J - \tilde{J} = \bar{J}$ is also a Jacobi field (linear ODE)

w/ $\bar{J}(0) = 0 = \bar{J}(\ell)$

Non-conjugate $\Rightarrow$ $\bar{J} \equiv 0$ on $[0,\ell]$.

$\Rightarrow$ $J = \tilde{J}$ on $[0,\ell]$.

$\boxed{\text{Existence:}}$ $\dim \{ \underbrace{\text{Jacobi field } J : J(0) = 0}_{\Delta} \} = n$

$\Phi : \Delta \longrightarrow T_{\gamma(\ell)}M$ by $\Phi(J) = J(\ell)$

- Φ = linear map
- Φ = isomorphism because of rank thm.

(injective $\Leftarrow$ ($\Phi(J) = 0$ iff $J(\ell) = 0$ iff $J \equiv 0$))

$\Rightarrow$ $\forall\, w \in T_{\gamma(\ell)}M$, $\exists\, J_w \in \Delta$ s.t.

$$\Phi(J_w) = w = J(\ell)$$

$\therefore$ $\exists$ J field s.t. $\begin{cases} J(0) = 0 \\ J(\ell) = w. \end{cases}$

Similarly, $\exists\, \bar{J}$ s.t. $\bar{J}(l) = 0$, $\bar{J}(0) = v$.
(interchange $\gamma(0)$, $\gamma(l)$)

$$\tilde{J} = J + \bar{J} \text{ is a Jacobi field s.t.}$$

$$\tilde{J}(0) = v, \quad \tilde{J}(l) = w. \quad \#.$$

Recall: If $\alpha : \overset{t}{[a,b]} \times \overset{v}{(-\varepsilon, \varepsilon)} \times \overset{w}{(-\delta, \delta)} \longrightarrow M$ s.t.
(variation of geodesic)

$$\begin{cases} \alpha(a, v, w) = p \\ \alpha(b, v, w) = q \end{cases}$$ And $\alpha(\cdot, 0, 0) =$ Normal geodesic

then $$\frac{\partial^2 L}{\partial v \partial w} = \int_a^b \langle \nabla_T V, \nabla_T W \rangle + R(W, T, V, T)\, dt$$

Defn: $\forall\, V, W$ along $\gamma(t)$,

Define $$I(V, W) = \int_a^b \langle \nabla_T V, \nabla_T W \rangle + R(W, T, V, T)\, dt$$

called the index form.

prop: Let I be defined on all piecewise smooth vector fields along the geodesic which vanished at end pts a, b. Then the Null space of I = set of Jacobi field which vanishes at a, b.

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pf: Let $V \in$ Null space of I vanishing at a, b

$$I(V, W) = 0 \quad \forall \ W \text{ piecewise smooth}, \ W(a) = W(b) = 0$$

Since V = piecewise smooth, $\exists \ a = t_0 < t_1 < \cdots < t_k = b$

s.t. $V|_{(t_i, t_{i+1})}$ is smooth.

On each (t_i, t_{i+1}), let f be smooth fcn

s.t. $f(t_i) = f(t_{i+1}) = 0$, vanishing outside $[t_i, t_{i+1}]$

and $W = f(t)\left(-\nabla_T \nabla_T V + R(T, V)T\right)$

$$T\langle \nabla_T V, W\rangle - \langle \nabla_T \nabla_T V, W\rangle$$

$$0 = I(V, W) = \int_{t_i}^{t_{i+1}} \langle \nabla_T V, \nabla_T W\rangle + R(W, T, V, T)$$

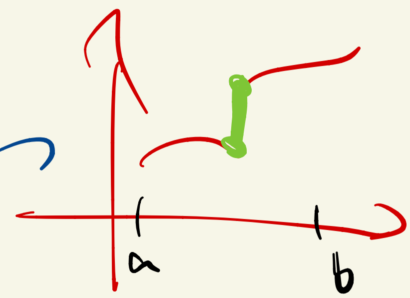
$\left(f=0 \text{ at } t_i, t_{i+1}\right)$

$$= \int_{t_i}^{t_{i+1}} \langle W, -\nabla_T \nabla_T V + R(T,V)T \rangle$$

$$0 = \int_{t_i}^{t_{i+1}} f(t) \cdot \left\| -\nabla_T \nabla_T V + R(T,V)T \right\|^2$$

$\Rightarrow$ V is Jacobi field on each sub-interval.

$\because$ V = piecewise Jacobi field.



$$\therefore \underset{\overset{\|}{0}}{I(V,W)} = \sum_{i=1}^{n} \left\langle \underline{\left(\lim_{t \to t_i^+} - \lim_{t \to t_i^-} \right)} \nabla_T V, W \right\rangle$$

$\Rightarrow$ V don't have corner if we choose W s.t.

$$W = \underline{\left(\lim_{t \to t_i^+} - \lim_{t \to t_i^-} \right)} \nabla_T V \quad \text{at } t_i$$

$\#$.

Index lemma : (J-field $\boxed{\text{minimize}}$ Index form)

If γ is a geodesic from p to q s.t. there are no conjugate pt. of p along γ. Let V be the unique Jacobi field s.t. $V(p) = W(p) = 0$ and

$$V(q) = W(q)$$

Exists thanks to conj. pt

then we have $\quad I(V,V) \leq I(W,W)$.

Moreover, Equality holds iff $V = W$.

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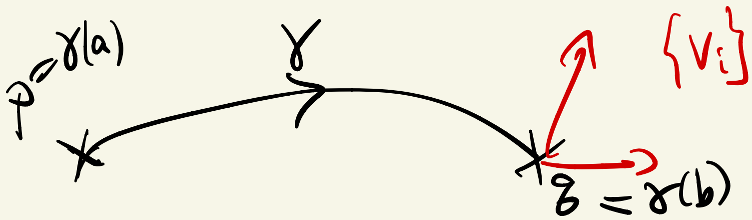
pf: At $q \in M$, let $\{V_i\}_{i=1}^{n}$ be a basis of T_qM.



By Assumptions on conjugate pts, $\exists$ J-field $V_i(t)$ along $\gamma(t)$ s.t. $V_i(p) = 0$, $V_i(q) = V_i$, $\forall i = 1, 2, \cdots n$

$$\Rightarrow \quad V_i(t) = t A_i(t) \quad \text{for some smooth } A_i(t)$$

$(\gamma : [a,b] \to M)$

then $\{A_i(t)\}$ is linearly indep $\forall t \in [a,b]$

For a smooth VF. W s.t. $W(a) = W(p) = 0$,

we have $\quad W(t) = \sum_{i=1}^{n} g_i(t) A_i(t)$ for some $g_i(t)$

$$= \sum_{i=1}^{n} f_i(t)\, V_i(t).$$

compute $I(V,V)$ and $I(W,W)$:

V = Jacobi field s.t. $V(q) = W(q)$

$$\Rightarrow V = \sum_{i=1}^{n} f_i(b) \cdot V_i(t). \quad (\text{by uniqueness})$$

$$I(V,V) = \int_a^b \langle V', V' \rangle + R(T, V, T, V)$$

J-field

$$= \langle V'(q), V(q) \rangle$$

$$= f_i(b)\, f_j(b) \langle V_i'(b), V_j(b) \rangle \ \#.$$

$$I(W,W) = \int_a^b \langle \boxed{W'}, W' \rangle + R(T, W, T, W)$$

$$W' = \nabla_T W = \nabla_T \left(\sum f_i \cdot V_i \right) = f_i' V_i + f_i V_i'$$

$$= \int_a^b 2 f_i' f_j \langle V_i, V_j' \rangle + f_i' f_j' \langle V_i, V_j \rangle$$

$$+ f_i f_j \langle V_i', V_j' \rangle + \boxed{f_i f_j R(T, V_i, T, V_j)}$$

$$\int_a^b f_i f_j \langle V_i', V_j' \rangle$$

$$= \int_a^b \partial_t \left(f_i f_j \langle V_i, V_j' \rangle \right) - f_i' f_j \langle V_i, V_j' \rangle$$

$$- \langle f_i V_i, (f_j V_j')' \rangle$$

$I(V_i V)$

$(V_i(a) = 0)$

$$= \boxed{f_i(b) f_j(b) \langle V_i(b), V_j'(b) \rangle} - \int_a^b \langle f_i' V_i, f_j V_j' \rangle$$

$$- \int_a^b \langle f_i V_i, f_j' V_j' + \boxed{\cancel{f_j V_j''}} \rangle \quad \#$$

?? w/ $f_i f_j R(T, V_i, T, V_j)$ using J-field.

$$\Rightarrow I(W, W) = I(V, V) + \int_a^b \boxed{f_i' f_j' \langle V_i, V_j \rangle} \geq 0$$

$$+ \int_a^b \underbrace{f_i' f_j \langle V_i, V_j' \rangle - f_i' f_j \langle V_i', V_j \rangle}_{= 0 \text{ why??}}$$

Suffices to show that $\langle V_i, V_j' \rangle = \langle V_i', V_j \rangle, \forall i, j.$

$$\left(\langle V_i', V_j\rangle - \langle V_i, V_j'\rangle\right)'$$

$$= \langle V_i'', V_j\rangle + \cancel{\langle V_i', V_j'\rangle} - \cancel{\langle V_i', V_j'\rangle} - \langle V_i, V_j''\rangle$$

$$= -\langle R(V_i'', \gamma')\gamma', V_j\rangle + \langle R(V_j'', \gamma')\gamma', V_i\rangle$$

$$= -R(V_i'', \gamma', \gamma', V_j) + R(V_j'', \gamma', \gamma', V_i)$$

$= 0$. (by symmetry of Rm) #

at $t=a$

$$\therefore \langle V_i', V_j\rangle - \langle V_i, V_j'\rangle = \text{constant} = 0 \quad \#.$$

$$\boxed{\therefore I(V,V) + \int_a^b f_i' f_j' \overset{\geq 0}{\langle V_i, V_j\rangle} = I(W,W)}$$

$\forall\ W(q) = V(q)$, $W(p) = V(p) = 0$. (smooth)

If Equality holds, then $f_i'(t)\, V_i(t) \equiv 0$

If $V_i(t_0) = 0$ for some $t_0 \in (a,b)$

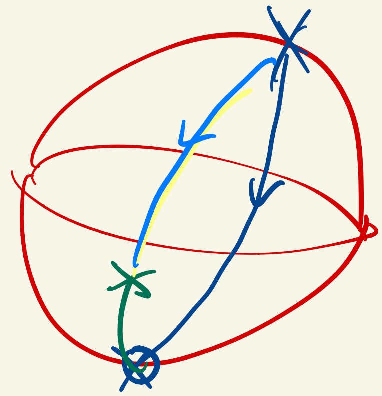
then $V_i \equiv 0$ on $[a, t_0] \Rightarrow V_i \equiv 0$ on $[a,b]$

By conjugate   uniqueness

But this is impossible as $V_i(b) \neq 0$.

$\therefore f'(t) \equiv 0 \Rightarrow \boxed{V = W}$ #

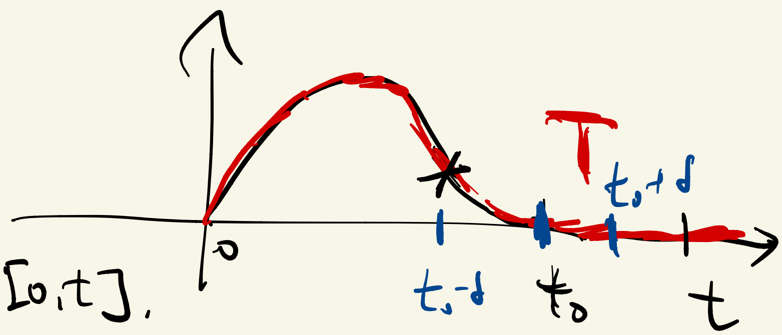
Corollary: Let $\gamma: [0, +\infty) \to M$ be a geodesic. If $\gamma(t_0)$ = conjugate to $\gamma(0)$, then $\gamma|_{[0,t]}$ is not minimizing geodesic for all $t > t_0$.



pf: We may assume t_0 to be the first conjugate pt.

Let J be a Jacobi field along $\gamma|_{[0,t_0]}$ s.t.

$$J(0) = 0 = J(t_0)$$



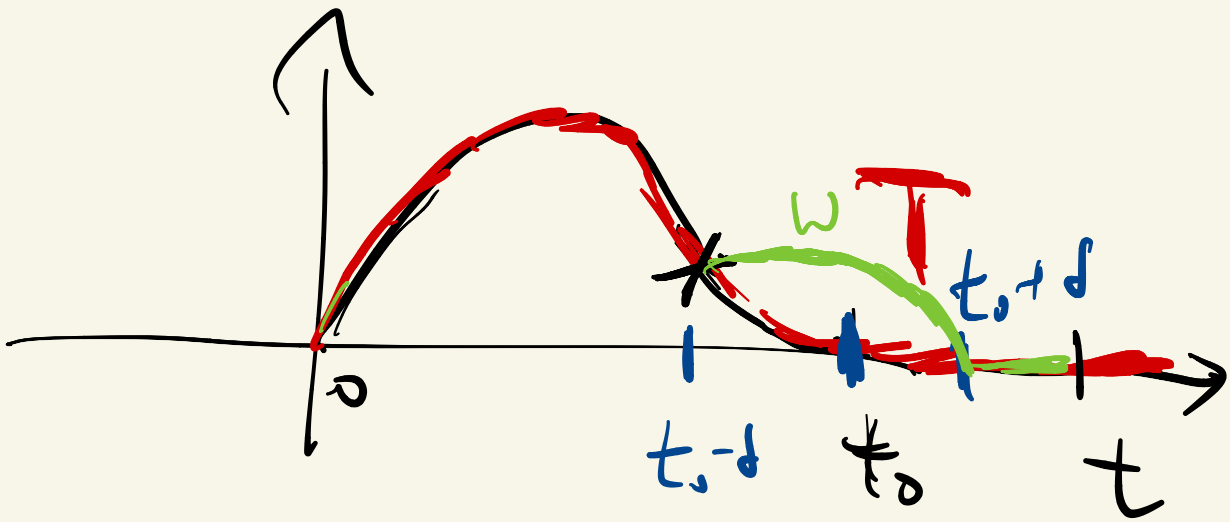
For the index form on $[0,t]$,

$I(T,T) = 0$ $\left(\begin{array}{l}\text{on } [0,t_0], \text{ given by } J\text{-field} \\ \text{on } [t_0,t] \text{ given by zero}\end{array}\right)$

Choose $\delta \ll 1$ s.t. $\gamma(t_0+\delta)$ is Not conjugate to $\gamma(t_0-\delta)$ (exp is not singular if $\delta \ll 1$)

$\therefore \exists$ Jacobi field W on $\gamma|_{[t_0-\delta, t_0+\delta]}$ s.t.

$$\begin{cases} W(t_0-\delta) = J(t_0-\delta) \\ W(t_0+\delta) = 0 \end{cases}$$



Define
$$X = \begin{cases} J & \text{on } [0, t_0-\delta] \\ W & \text{on } [t_0-\delta, t_0+\delta] \end{cases}$$

then $I(X,X) < 0$ since

$$I(X,X) = \left(\int_0^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^{t}\right)\left[|\nabla_{\dot\gamma} X|^2 + R(X,\dot\gamma,\dot\gamma,X)\right]$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

- $\underline{II} = 0$ as $X = 0$ on $[t_0+\delta, t]$

- $I = \int_0^{t_0-\delta} |\nabla_\gamma T|^2 + R(\gamma', T, \gamma', T)$ since $J = T$.

- $\underline{II} = \int_{t_0-\delta}^{t_0+\delta} |\nabla_{\gamma'} W|^2 + R(\gamma', W, \gamma', W)$

$= $ Index form on $\gamma|_{[t_0-\delta, t_0+\delta]}$

<u>Since</u>: $W =$ Jacobi field w/ $\begin{cases} W(t_0+\delta) = 0 = T(t_0+\delta) \\ W(t_0-\delta) = T(t_0-\delta) = J(t_0-\delta) \end{cases}$

Index lemma $\Rightarrow$ $\underline{II} < \int_{t_0-\delta}^{t_0+\delta} |\nabla_{\gamma'} T|^2 + R(\gamma', T, \gamma', T)$

since T is NOT smooth at t_0, $T \neq W$

by Equality of index lemma

$\Rightarrow I(X, X) < I(T, T) = 0$ $\#$

If $\gamma|_{[a,t]}$ is minimal, then for all variational vector field which is vanishing at end pts, we have $\left.\frac{\partial^2 L}{\partial s^2}\right|_{s=0} \geq 0$.

which contradicts with $I(X,X) < 0$ if we choose X to be variational V.F.

(do-able since $X(0) = X(t) = 0$)