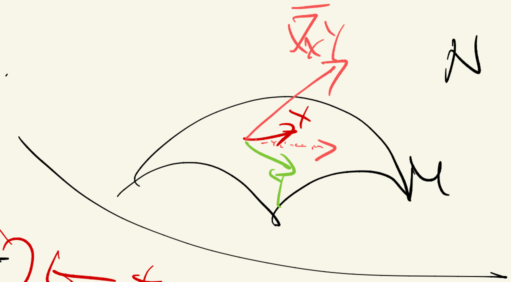
Let $M \subset N$ be a submfd.




Define the connection on $M$:

- $\forall X, Y \in \Gamma(TM)$, $\nabla_X Y = (\bar{\nabla}_X Y)^T$ ← tangent plane

  where $\bar{\nabla}$ is the connection of $(N, h)$

- check if $\nabla^1$ define a connection on $M$ (ex.)

- More generally, think of $M \subset N$ as

  Embedding $M \overset{i}{\hookrightarrow} N$, $g = i^* h$ where $h$ is metric on $N$.

  s.t. $(M, g)$ is a abstract Riemannian mfd.

  then define the $\nabla^2$: Levi-Civita connection of $g$

  ☆☆: $\nabla^2 = \nabla^1$

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Goal: Relate Riemannian geom of $M$ w/ that of $N$.

<u>Defn</u>: $\forall X, Y \in \Gamma(TM)$, define 2nd fundamental form

☆ $\vec{A}(X, Y) = -(\bar{\nabla}_X Y)^n$ ← normal part. $= -\bar{\nabla}_X Y + \nabla_X Y$

vector.

Lemma: $\vec{A}$ is tensor on $M$ and is symmetric.

<u>pf</u>: Suffice to check: $\forall f \in C^\infty(M)$,

$$A(fX, Y) = f A(X, Y).$$

locally, $\vec{h} = \vec{h}_{ij}\, dx^i \otimes dx^j$.

$A(fX,Y) = -(\bar\nabla_{fX}Y)^n = -(f\cdot\bar\nabla_X Y)^n = f\ A(X,Y)$ #

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$A(X,Y) - A(Y,X) = (\bar\nabla_Y X)^n - (\bar\nabla_X Y)^n$ ← normal part

$= (\bar\nabla_Y X - \bar\nabla_X Y)^n = ([Y,X])^n$

$= 0$ since $X,Y \in \Gamma(TM)$ $(\Rightarrow [X,Y] \in \Gamma(TM))$

$\Downarrow$

$A(X,fY) = A(fY,X) = f A(Y,X) = f A(X,Y)$ #

In the following, I will use the general tensor computation:

For an immersion $F: M \to (N,h)$, $(M,g) \triangleq (M, F^*h)$

locally $\{x^i\}$ : local coord. of $M$

$\{y^\alpha\}$ : local coord. of $N$.

- $g_{ij} = F_i^\alpha F_j^\beta h_{\alpha\beta}$ locally. $F(\underbrace{x^1,\cdots,x^n}_{M}) = (\underbrace{y^1(x),\cdots,y^m(x)}_{N})$.

- $DF: TM \to FN \rightsquigarrow DF \in \Gamma(\underbrace{F^*TN}_{\text{pull-back bundle}} \otimes T^*M)$

  (locally, $DF = F_i^\alpha\, dx^i \otimes \frac{\partial}{\partial y^\alpha}$)

- Extend $\nabla$ on $F^*TN$ by $\nabla_X V = \bar\nabla_{dF(X)} V$.

  for $X \in \Gamma(TM)$, $V \in F^*TN$

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For Reference, Refer to "Curvature Problems" by C. Gerhardt.

Define: $\vec{H} = \mathrm{tr}_g A = g^{ij} A_{ij}^{\alpha} \frac{\partial}{\partial y^{\alpha}} \in F^*(TN)$ ; mean curvature vector.

why care ??

Lemma: If $\vec{T}$ is a compactly supported variational vector field on $M$.

then $$\frac{d}{dt}\Big|_{t=0} A(M_t) = \int_M \langle \vec{T}, \vec{H} \rangle d\mu_0 .$$

$\star$ If $\vec{H}=0$ on $\Sigma$, then the area functional is a local minimum

pf: $F_t : M \hookrightarrow N$, $g(t) = F_t^* h$ is varying set. $\partial_t F\big|_{t=0} = \vec{T}$.

$$A(M_t) = \int_M d\mu_t \quad \text{where} \quad d\mu_t = \sqrt{\det g_t}\, dx .$$ (volume form for Riemannian mfld)

$$\frac{d}{dt} A(M_t) = \int_M \left( \frac{d}{dt} \sqrt{\det g} \right) dx = \int_M \frac{1}{2\sqrt{\det g}} \left( \frac{\partial}{\partial t} \det g \right) dx$$

Jacobi formula (wiki)
$$\frac{d}{dt} \det A = \det A \cdot \mathrm{tr}(A^{-1} A')$$

$$= \int_M \frac{1}{2} \sqrt{\det g} \cdot \mathrm{tr}_g(g')\, dx$$

$$= \int_M \frac{1}{2} \mathrm{tr}_g g' \cdot d\mu_t .$$

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$$F_{ti}^{\alpha} = (F_t)_i^{\alpha} = T_i^{\alpha}$$

$$\partial_t g_{ij} = \partial_t [F_i^{\alpha} F_j^{\beta} h_{\alpha\beta}] = F_{ti}^{\alpha} F_j^{\beta} h_{\alpha\beta} + F_i^{\alpha} F_{tj}^{\beta} h_{\alpha\beta} + F_i^{\alpha} F_j^{\beta} \partial_t h_{\alpha\beta}$$

$$= T_i^{\alpha} F_j^{\beta} h_{\alpha\beta} + F_i^{\alpha} T_j^{\beta} h_{\alpha\beta} + F_i^{\alpha} F_j^{\beta} \partial_{\gamma} h_{\alpha\beta} \cdot F_t^{\gamma} .$$

choose Normal coord at $F(p)$

$$\overset{\star}{=} \nabla_i T^{\alpha} \cdot F_j^{\beta} h_{\alpha\beta} + F_i^{\alpha} \nabla_j T^{\beta} h_{\alpha\beta} + \cancel{0} .$$ ($\partial_{\gamma} h_{\alpha\beta} = 0$ in Normal coord)

$$\Rightarrow \frac{1}{2} g^{ij} g'_{ij} = g^{ij} \nabla_i T^\alpha \cdot F^\beta_j h_{\alpha\beta}$$

$$\therefore \left.\frac{d}{dt}\right|_{t=0} A(M_t) = \int_M g^{ij} T^\alpha_i F^\beta_j h_{\alpha\beta} \, d\mu_0$$

$$\overset{T\in C_c^\infty,\ \text{stoke}}{=} \int_M - g^{ij} T^\alpha F^\beta_{ij} h_{\alpha\beta} \, d\mu_0$$

$$\overset{*}{=} \int_M \langle \vec{T}, \vec{H} \rangle \, d\mu_0$$

Claim: $F^\alpha_{ij} = A^\alpha_{ij}$.

locally, $F^\alpha_{ij} = \partial_i\partial_j F^\alpha - \Gamma^k_{ij} \underbrace{\partial_k F^\alpha}_{F^\alpha_k} + \bar{\Gamma}^\alpha_{\beta\gamma} F^\beta_i F^\gamma_j$

If $M^n \subset N^m$, we choose coord s.t.

$(\underbrace{x^1, \dots, x^n}_{\text{coord. of } M}, y^{n+1}, \dots, y^m)$ is coordinate of $N$

then $F^\alpha_i = \delta^\alpha_i$ and $F^\alpha_{ij} = \bar{\Gamma}^\alpha_{ij} - \Gamma^\alpha_{ij}$ = Normal part of $\bar{\nabla}$.

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2nd variational formula (for minimal ($\vec{H}=0$) hypersurface with $\vec{T} = f\nu$, $f \in C_c^\infty$).

(Sometimes ppl use $\vec{H} = H\nu$) mean curvature

$$\left.\frac{d^2}{dt^2}\right|_{t=0} A(M_t) \ (= \ ??) \quad \text{if } \vec{H} = 0 \text{ at } t=0.$$

$$\| $$

$$\left.\frac{d}{dt}\right|_{t=0} \left( \int_M \langle \vec{H}, \vec{T} \rangle \, d\mu_t \right)$$

$$= \int_M \left( \langle \partial_t \vec{H}, \vec{T} \rangle + \langle \vec{H}, \partial_t \vec{T} \rangle \right) d\mu_t + \langle \vec{H}, \vec{T} \rangle (d\mu_t)'$$

$(H=0)$ $$= \int_M \langle \partial_t (H\nu), f\nu \rangle \, d\mu_0 \qquad \vec{T} = f\nu.$$

$$= \int_M f H' \, d\mu_0$$

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$$H' = (A_{ij} g^{ij})' \quad \text{where} \quad A_{ij} = \langle \vec{A}_{ij}, \nu \rangle = -\langle \overline{\nabla}_{F_i} F_j, \nu \rangle$$

where $F_i = dF(\partial_i)$.

$$= (g^{ij})' A_{ij} + g^{ij} A_{ij}'$$

$$= -g^{ip} g^{jq} \partial_t g_{pq} A_{ij} + g^{ij} (A_{ij})'$$

$$\partial_t (g^{ij} g_{jk}) = (\delta^i_k)' = 0$$

$$\rightsquigarrow \partial_t g^{ij} = -g^{ip} g^{jq} \partial_t g_{pq}$$

$$= \mathrm{I} + \mathrm{II}.$$

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$$\mathrm{I} := -2 g^{ip} g^{jq} ( F^\alpha_{pt} F^\beta_q h_{\alpha\beta}) A_{ij}$$

$$= -2 g^{ip} g^{jq} F^\beta_q h_{\alpha\beta} A_{ij} (f\nu)^\alpha_p \qquad \partial_t F = f\nu$$

$$= -2 g^{ip} g^{jq} F^\beta_q h_{\alpha\beta} A_{ij} (f_p \nu^\alpha + f \nu^\alpha_p)$$

$$= -2f g^{ip} g^{jq} F^\beta_q h_{\alpha\beta} A_{ij} \nu^\alpha_p \qquad \Big( F^\beta_q \nu^\alpha h_{\alpha\beta} = \langle \underbrace{F_q}_{\text{tangent to } F(M)}, \nu \rangle = 0 \Big)$$

$$\nu_p = \overline{\nabla}_p \nu \perp \nu \Rightarrow \text{“} \overline{\nabla}_p \nu \in TM \text{”}$$

$$\overline{\nabla}_p \nu = \langle \overline{\nabla}_p \nu, e_q \rangle e_q \quad \text{if } \{e_i\} \text{ is o.n. to } F(M).$$

$$= -\langle \nu, \overline{\nabla}_p e_q \rangle = A(e_p, e_q)$$

$$\Rightarrow \nu_p = A_{pq} g^{qk} \partial_k = A^q_{\cdot p} \partial_q \text{ on } TM. \quad (A^q_p = g^{q\ell} A_{p\ell})$$

$$= -2f\, g^{ip} g^{jq}\, P^\beta_q\, h_{\alpha\beta}\, A_{ij}\, A^r_p\, F^\alpha_r$$

$\nu^\alpha_p = A^q_p\, dF(\partial_q) = A^q_p F^\alpha_q$

$$= -2f\, g^{ip} g^{jq}\, \cancel{g_{rq}}\, A_{ij}\, A^{\cancel{r}}_{pq}$$

$$= -2f\,|A|^2$$

abuse of Notations.

$$\text{II} = g^{ij}(A_{ij})' = -g^{ij}\langle \bar\nabla_i \partial_j, \nu\rangle'$$

" $\partial_i : dF(\partial_i)$ "

$$= -g^{ij}\langle \bar\nabla_i \partial_j, \nu'\rangle - \underline{g^{ij}\langle \bar\nabla_t \bar\nabla_i \partial_j, \nu\rangle}$$

$$= \text{III} + \text{IV}$$

$$\text{IV} = -g^{ij}\langle \bar\nabla_i \bar\nabla_j \partial_t, \nu\rangle - g^{ij}\bar R_{tij\nu}$$

$$= -g^{ij}\langle \bar\nabla_i \bar\nabla_j (f\nu), \nu\rangle - f\bar R_{\nu\nu}$$

$\langle \bar\nabla_i \partial_j, \nu'\rangle$
$\nu' \in TM$
$= \langle \nabla_i \partial_j, \nu'\rangle$

$\text{III} = 0$ seen by choosing Normal coordinate at $p$

$$\therefore \left.\frac{d^2}{dt^2}\right|_{t=0} A(M_t) = \int_M -2f^2|A|^2 - f^2\bar R_{\nu\nu} - \underline{f g^{ij}\langle \bar\nabla_i \bar\nabla_j (f\nu), \nu\rangle}\, d\mu_0$$

Claim: $\displaystyle\int_M -g^{ij}\langle \bar\nabla_i \bar\nabla_j (f\nu), f\nu\rangle\, d\mu_0 \overset{\text{Goal}}{=} \boxed{\int_M f^2|A|^2\, d\mu_0 + \int_M |\nabla f|^2\, d\mu_0}$

$$\| $$

$$\int_M \underbrace{|\bar\nabla_i (f\nu)|^2}_{\text{cf Normal coordinate}}\, d\mu_0 \;/\!/\; \int_M f^2|\bar\nabla \nu|^2 + \int_M |\nabla f|^2\, d\mu_0$$

$\nu_i = A^j_i F_j$

$$= \int_M f^2|A|^2 + \int_M |\nabla f|^2\, d\mu_0$$

Under minimal hypersurface, we have

$$\boxed{\therefore \ \frac{d^2}{dt^2}\Big|_{t=0} A(M_t) = \int_M -(|A|^2 + \overline{Ric}(\nu,\nu)) f^2 + |\nabla f|^2 \, d\mu.}$$

if the variational vector field is given by $f\nu$.

See P. Li Ch. 1 using $\{e_i\}$ approach.

Coro: If $M^n$ is cpt and is minimal in $N^{n+1}$ with $\overline{Ric}(\nu) > 0$, then $M$ cannot be stable minimal.

pf: If $M$ is stable minimal, then $\forall f \in C^\infty(M)$,

$$0 \le \frac{d^2}{dt^2}\Big|_{t=0} A(M_t) = -\int_M (|A|^2 + \overline{Ric}(\nu,\nu)) f^2 + |\nabla f|^2$$

take $f \equiv 1 \Rightarrow \int_M |A|^2 < \int_M |A|^2 + \overline{Ric}(\nu,\nu) \le 0.$

$\times$

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Gauss equation: Let $X, Y, Z, W \in TM \subset TN$

(In general, modify by push forward)

tensor

$$\overline{R}(X,Y,Z,W) = \underbrace{R(X,Y,Z,W)}_{\text{tensor}} - \underbrace{\langle A(X,W), A(Y,Z)\rangle}_{\text{tensor}} + \langle A(Y,W), A(X,Z)\rangle.$$

pf: $\overline{R}(X,Y,Z,W) = \langle \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, W\rangle$

$$= \langle \bar{\nabla}_X (\bar{\nabla}_Y Z - A(Y,Z)) - \bar{\nabla}_Y (\bar{\nabla}_X Z - A(X,Z))$$

$$- \bar{\nabla}_{[X,Y]} Z + A([X,Y], Z), \ W \rangle$$

$$= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle + \langle A([X,Y], Z), W \rangle$$

$$- \langle A(\nabla_Y Z, X), W \rangle - \langle \bar{\nabla}_X A(Y,Z), W \rangle$$

$$+ \langle A(\nabla_X Z, Y), W \rangle + \langle \bar{\nabla}_Y A(X,Z), W \rangle$$

$$= R(X,Y,Z,W) - \langle \bar{\nabla}_X A(Y,Z), W \rangle + \langle \bar{\nabla}_Y A(X,Z), W \rangle$$

$$+ \langle A([X,Y], \cancel{Z}), W \rangle - \langle A(\cancel{\nabla_Y Z}, X), W \rangle + \langle A(\cancel{\nabla_X Z}, Y), W \rangle$$

Suffices to check the equality under Normal coordinate at p.

$\nabla_i \partial_j = 0$

then suffices to show.

$$\bar{R}_{ijkl} \overset{\checkmark}{=} R_{ijkl} - \langle \bar{\nabla}_i \vec{A}_{jk}, \partial_l \rangle + \langle \bar{\nabla}_j \vec{A}_{ik}, \partial_l \rangle$$

$$\boxed{= R_{ijkl} - \langle \vec{A}_{il}, \vec{A}_{jk} \rangle + \langle \vec{A}_{ik}, \vec{A}_{jl} \rangle}$$

Verify: $\vec{A}_{jk} = \sum_\alpha \langle \vec{A}_{jk}, \nu^\alpha \rangle \nu^\alpha$ where $\{\nu^\alpha\}$ are set of Normal v.F. of M.

$$\langle \partial_l, \bar{\nabla}_i \vec{A}_{jk} \rangle = \langle \sum_\alpha \bar{\nabla}_i (\langle \vec{A}_{jk}, \nu^\alpha \rangle \nu^\alpha), \partial_l \rangle$$

$$= \langle \sum_\alpha \partial_i (\langle \vec{A}_{jk}, \nu^\alpha \rangle) \nu^\alpha + \langle \vec{A}_{jk}, \nu^\alpha \rangle \bar{\nabla}_i \nu^\alpha, \partial_l \rangle$$

$$= -\sum_\alpha \langle \vec{A}_{jk}, \nu^\alpha \rangle \langle \nu^\alpha, \bar{\nabla}_i \partial_l \rangle$$

$$= \sum_\alpha \langle \vec{A}_{jk}, \nu^\alpha \rangle \langle \nu^\alpha, \vec{A}_{il} \rangle$$

$$= \langle \vec{A}_{jk}, \vec{A}_{il} \rangle . \quad \#$$

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Thm (Schoen-Yau) 3-Torus cannot admit smooth metric $g$ with $R(g) = g^{ij} R_{ij} > 0$.

( SY methods works for $n \leq 7$, $T^n$ )

( For general $n$, same thm holds by Gromov-Lawson )

Moreover, if $R(g) \geq 0$ then $Rm \equiv 0$.

sketch of pf:

$\boxed{\text{Some fact from topology: } T^3 \text{ doesn't contain sphere as closed sub-mfd.}}$

Let $[\alpha]$ be a non-trivial homology class in $T^3$

$[\alpha]$

• Find $\Sigma^2 \in [\alpha]$ s.t. $\Sigma$ has minimal area.

By GMT $\Rightarrow$ $\Sigma^2$ exists, and is stable.

2nd variation $\Rightarrow$ $\forall f \in C^\infty(\Sigma^2)$ $(f=1)$

$$0 \le \frac{d^2}{dt^2}\Big|_{t=0} A(\Sigma_t) = \int_\Sigma -(|A|^2 + \overline{R}_{\nu\nu})$$

$$\Rightarrow \int_\Sigma |A|^2 + \overline{Ric}(\nu,\nu) \le 0$$

$\|$ Gauss equ.

$$\int_\Sigma \cancel{|A|^2} + \frac{1}{2}\left(\overline{R}_M - R_\Sigma - \cancel{|A|^2} - \cancel{\overset{0}{H^2}}\right)$$

minimal

$\geq$

$$\int_\Sigma \frac{1}{2}(\overline{R} - R_\Sigma) \overset{(\overline{R}>0)}{>} -\int_\Sigma K_\Sigma$$

$$\Rightarrow \int_\Sigma K_\Sigma > 0 \overset{\text{Gauss-Bonnet}}{\Longrightarrow} \Sigma \cong S^2$$

$\boxed{\text{By topo. Result}}$

It remains to show that $R \geq 0 \Rightarrow Rm = 0$.

Claim: $R \geq 0 \Rightarrow Rc = 0$ on $T^3$ ($\because$ 3D $\Rightarrow Rm = 0$)

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Lemma: If $\partial_t g = h$, then

$$\partial_t R = -h^{ij} R_{ij} + \underline{\mathrm{div}(V)} \quad \text{for some vector field } V$$

$\mathrm{div}(\alpha) = \nabla_i \alpha^i$

pf:
$$\partial_t R = \partial_t (g^{ij} R_{ij}) = (g^{ij})' R_{ij} + g^{ij} R_{ij}'$$

$$= -g^{ip} g^{jq} h_{pq} R_{ij} + g^{ij} R_{ij}'$$

$$= -h^{ij} R_{ij} + \boxed{g^{ij} R_{ij}'}$$

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$$R_{ij}' = \underbrace{\partial_t R_{ij}}_{\text{tensor}} = \partial_t \left( \partial_l \Gamma_{ij}^l - \partial_i \Gamma_{lj}^l + \Gamma * \Gamma \right)$$

$$= \partial_l \underbrace{\boxed{\partial_t \Gamma_{ij}^l}}_{\text{tensor}} - \partial_i \partial_t \Gamma_{lj}^l + \partial_t \Gamma * \Gamma$$

Rmk: $\Gamma_{ij}^k \neq$ tensor

But $\Gamma_{ij}^k - \bar{\Gamma}_{ij}^k$ is a tensor.

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$$= \nabla_l \partial_t \Gamma_{ij}^l - \nabla_i \partial_t \Gamma_{lj}^l \quad \text{(By changing Normal coord.)}$$

$$= \nabla_l \alpha_{ij}^{l} - \nabla_i \alpha_{lj}^{l}$$

$$\therefore \; d_t R = -h^{ij} R_{ij} + g^{ij} (\nabla_l \alpha_{ij}^{l} - \nabla_i \alpha_{lj}^{l})$$

$$= -h^{ij} R_{ij} + \nabla_l V^l - \cancel{\nabla_i \alpha_l^{li}} \; \nabla_l \alpha_i^{il}$$

$$= -h^{ij} R_{ij} + \nabla_l W^l.$$

where $W^l = g^{ij} \alpha_{ij}^{l} - \alpha_i^{il}$.

pf of Rigidity (Sketch):

Suppose $Ric(g) \not\equiv 0$.

Consider $g(t) = g - t\,Ric(g)$ smooth metric if $t \ll 1$.

Consider $\lambda_1(-f\Delta + R)$ : the first eigenvalues of operator $-f\Delta_{g(t)} + R_{g(t)} = L_{g(t)}$.

PDE $\Rightarrow \exists\, u(t)$ s.t. $\int_M u^2 = 1$, $u > 0$

and $-f\Delta u + Ru = \lambda_1 u$ on $M$.

If $\lambda_1(-\gamma\Delta + R) > 0$, then

conformal Laplacian :

$\tilde{g} = u^4 g$ satisfies $\tilde{R} = u^{-5}(-\gamma\Delta + R)u$

$= \lambda_1 u^{-4} > 0$

$\Rightarrow$ M supports PSC (positive scalar curvat.)

$\Rightarrow$ Impossible since $M = T^3$.

$\therefore \quad \lambda_1(g(t)) \le 0, \ \forall \, t \ll 1.$

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at $t=0$, $\lambda_1(g) \le 0$.

$\lambda_1(g) = \lambda_1(-\gamma\Delta + R) \geq 0$

Recall: $\lambda_1(L) = \inf\left\{ \int \langle Lu, u\rangle : \int u^2 = 1 \right\}$

$\boxed{\int |\nabla u|^2 + Ru^2 = \lambda_1 = 0, \ \boxed{u \not\equiv 0}}$

$\Rightarrow \lambda_1(g) = 0 \Rightarrow R_g = 0$ and $u(\cdot) \equiv \text{const}.$

MP

$\left( (-\gamma\Delta + R)u = 0 \right.$
$\left. u > 0. \right)$

$$\left.\frac{d}{dt}\right|_{t=0} \lambda_1(g(t)) = \left.\frac{d}{dt}\right|_{t=0} \int_M f\, |\nabla u|^2 + R u^2 \, d\mu$$

"pretend this is well-defined"

In general, using Dini-Derivative

some const.

$$= c_0 \int_M D_t R \, d\mu$$

$$= c_0 \left[ \int_M - h^{ij} R_{ij} + \operatorname{div}(W) \, d\mu \right]$$

$h = -Ric$

Stokes thm.

$$= c_0 \left( \int_M |Ric|^2 + \int_M \overset{0}{\cancel{\operatorname{div}(W)}} d\mu \right)$$

$$= \overset{>0}{c_0} \int_M \overset{>0}{|Ric|^2}$$

$(Ric \not\equiv 0.)$

$$\Rightarrow \begin{cases} \lambda_1(g(0)) = 0 \\ \left.\frac{d}{dt}\right|_{t=0} \lambda_1(g(t)) > 0 \end{cases}$$

$$\Rightarrow 0 \geq \lambda_1(g(t)) > 0 \quad \forall\, 0 < t \ll 1.$$

$\Rightarrow$ Impossible since otherwise

$M = T^3$ supports PSC metric.

$$\Rightarrow Ric(g) \equiv 0 \text{ on } T^3. \quad \#$$