Last time: introduce $\exp_p : \text{Domain} \subseteq T_pM \to M$

$\text{Domain} \supseteq B(\varepsilon)$

Normal coordinate around $p \in M$.

fix $\{e_i\}_{i=1}^n$ o.n. in T_pM.

* $\varphi(x^1, \cdots, x^n) = \exp_p\left(\sum_{i=1}^n x^i e_i\right) \in M$.

(i.e. $(x^1, \cdots, x^n) \in B(0,\varepsilon) \subseteq \mathbb{R}^n \longmapsto \varphi(x) \in M$.)

* satisfies $\Gamma_{ij}^k(p) = 0$ (or equivalently $\nabla_{\partial_i}\partial_j = 0$ $\forall i,j$ at p)

Example: Given $f: M \to \mathbb{R}$, smooth fcn on M.

$\nabla f \in \Gamma(TM)$ given by $\langle \nabla f, X\rangle = X(f)$, $\forall X \in \Gamma(TM)$

locally, $\nabla f = \sum_{i=1}^n f_i g^{ij} \frac{\partial}{\partial x^j}$ (dual to df wrt g)

$\nabla^2 f = \text{Hessian of } f = \nabla(\nabla f)$.

- In $\mathbb{R}^n$, $\nabla^2 f = (f_{ij})_{1\le i\le n, 1\le j\le n}$, where $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$

- For general Riemannian mfd, $\nabla^2 f = \sum_{i,j} \nabla_i \nabla_j f \cdot dx^i \otimes dx^j$

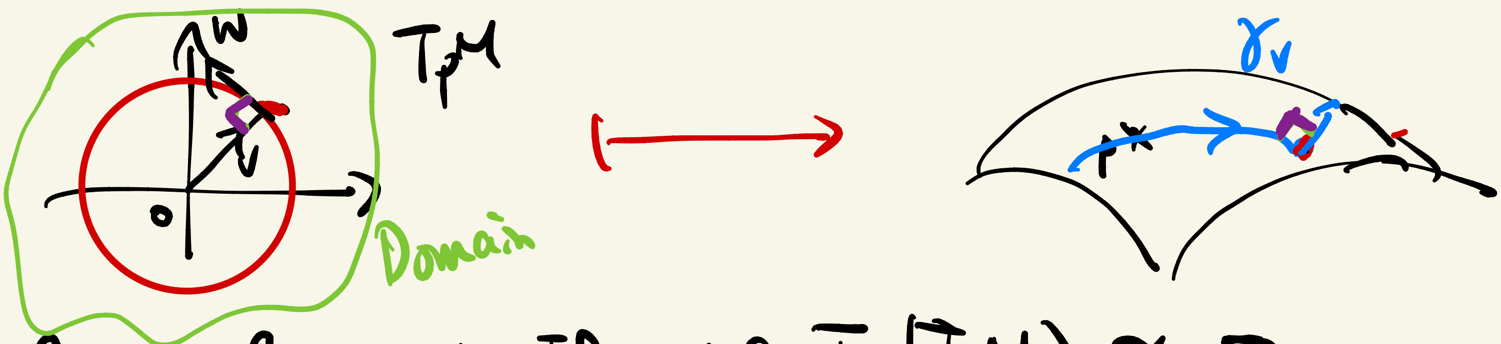
By def of $\nabla\alpha$: $\nabla_i\nabla_j f = (\nabla^2 f)(\partial_i, \partial_j)$

Def $\overset{\text{Def}}{=} \partial_i [(\nabla f)(\partial_j)] - (\nabla f)(\nabla_i \partial_j)$

$$= \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f.$$

But under Normal coordinate at p.

$$\nabla_i \nabla_j f = \partial_i \partial_j f. \quad \text{at } p.$$



Gauss lemma: If $w \in T_v(T_pM) \cong T_pM$ and assumed $\exp_p$ is well-defined on $B(\varepsilon) \subseteq T_pM$ where $|v| < \varepsilon$, then

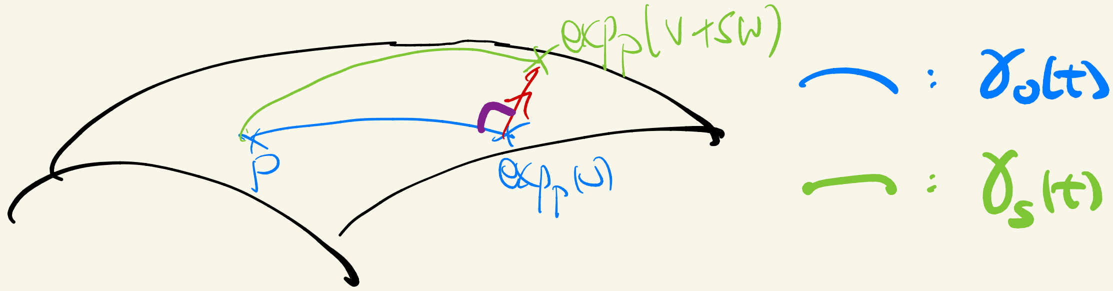
$$\langle d\exp_p|_v(v),\ d\exp_p|_v(w) \rangle_{g(\exp_p(v))} = \langle v, w \rangle_{g(p)}.$$

pf: Consider $\gamma_s(t) = \exp_p(t(v + sw))$, $0 \le t \le 1$, $|s| < \delta \ll 1$.

which is well-defined by stability of ODE.

$\gamma_s(t)$ = family of geodesic s.t.

• $\gamma_s(0) = p$ • $L(\gamma_s) = |v + sw|$ $\forall |s| < \delta$



(i) $\displaystyle \frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{d}{ds}\Big|_{s=0} |sw + v|_{g(p)} = \frac{1}{|v|} \langle v, w \rangle \Big|_p .$

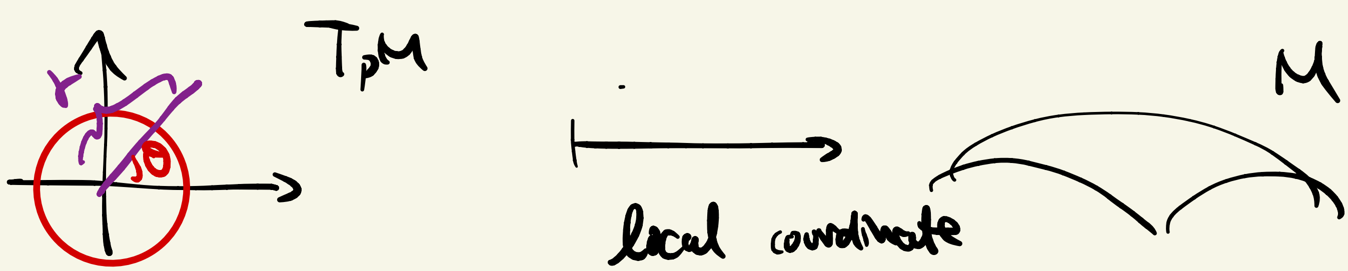
(ii) $\Big\|$ 1st variational formula

$$\frac{1}{|v|} \left\langle \frac{d}{ds}\Big|_{s=0} \gamma_s(1) , \frac{\partial}{\partial t} \gamma_0(1) \right\rangle_{g(\exp_p(v))} \quad (\because \gamma_0 = \text{geod.})$$

$$= \frac{1}{|v|} \left\langle d\exp_p\big|_v(w) , d\exp_p\big|_v(v) \right\rangle_{g(\exp_p(v))}$$

$$\therefore \langle v, w \rangle_{g(p)} = \left\langle d\exp_p\big|_v(v) , d\exp_p\big|_v(w) \right\rangle_{g(\exp_p(v))}$$

<u>Conseq.</u>: Let $(r, \theta^1, \ldots, \theta^{n-1})$ be the polar coordinate.



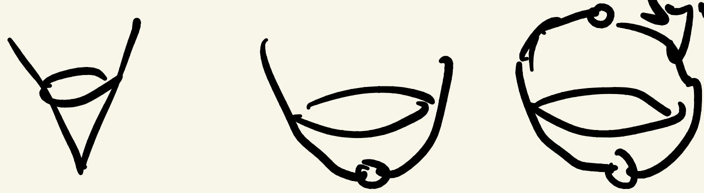
Express g using $dr, d\theta^i$ (away from p)

$$g = a_{11}\, dr\otimes dr + a_{ij}\, dr\otimes d\theta^j + a_{ij}\, d\theta^i\otimes d\theta^j$$

- $a_{11} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$ ($\because$ $\exp_p$ preserve length)
- $a_{ij} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^j}\right) = 0$ ($\because$ Gauss lemma)

$$\therefore\ g = dr\otimes dr + \underset{a_{ij}(r,\theta)}{\overset{\text{weighting}}{\boxed{a_{ij}}}}\, d\theta^i\otimes d\theta^j$$

Rmk: ① if a_{ij} = indep of θ, then g = Rotationally symmetric

Eg:  , ...

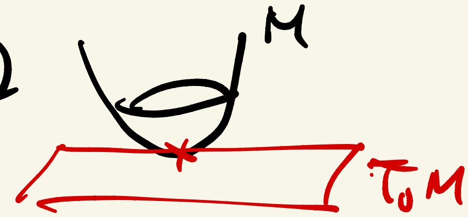
② $\nabla r = \frac{\partial}{\partial r}$ in normal coordinate

where $r(x) = \sqrt{\sum_{i=1}^{n} x_i^2}$ is local fcn

smooth wherever $\exp_p$ is smooth

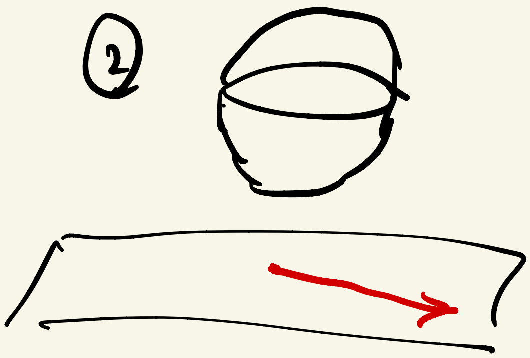
Q: when is exp defined on whole T_pM??

Example: ①

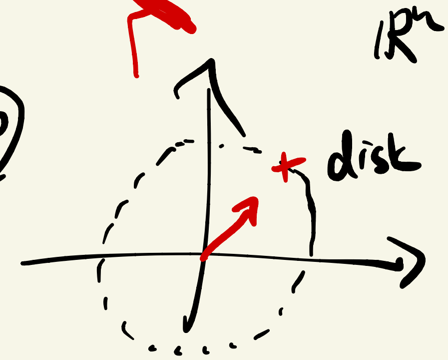


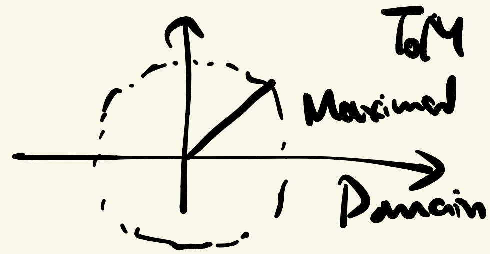
$\rightsquigarrow$ $\exp_p$ defined on T_pM

②



③





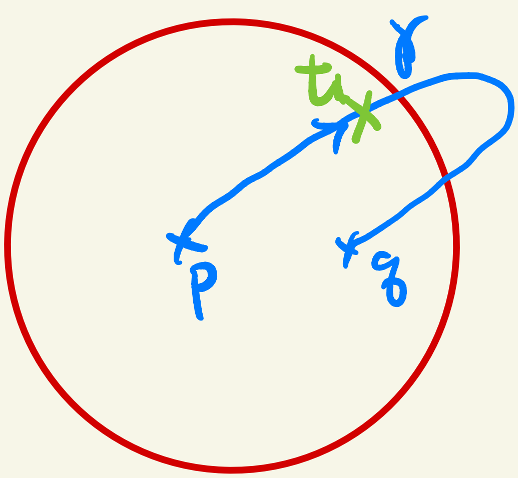
Corollary: ① If $\exp_p : B(\bar{r}) \to M$ is a local diff.

then $\forall\, v \in B(\bar{r}) \subseteq T_pM$, $\gamma_v : [0,1] \to M$ is the unique curve s.t. $L(\gamma) = d(p, \exp_p(v)) = |v|$

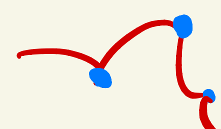
② If $d(\gamma(0), \gamma(1)) = L(\gamma)$ for some $\gamma : [0,1] \to M$

then γ is a smooth geodesic.

pf of ①:



Let γ be



a piecewise smooth curve from p to $q = \exp_p(v)$.

$$L(\gamma) = \int_0^1 |\gamma'|\, dt = \left(\int_0^{t_1} + \int_{t_1}^1\right) |\gamma'|\, dt$$

$\because$ on $B(\bar{r})$, $r(x) = \sqrt{\sum_{i=1}^{n} x_i^2}$ is smooth

$$\int_0^{t_1} |\gamma'|\, dt \geq \int_0^{t_1} \langle \gamma', \nabla r \rangle\, dt \quad (\because |\nabla r| = 1)$$

$$= r(\gamma(t_1))$$

① if γ intersect $\partial B(\bar{r})$, then may choose $t_1 > 0$ s.t. $r(\gamma(t_1)) = |v| < \bar{r}$

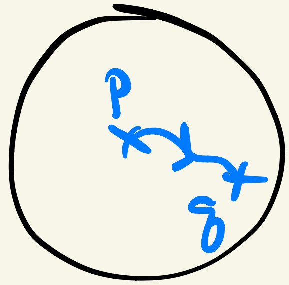
$$\Rightarrow L(\gamma) = |v| + \int_{t_1}^1 |\gamma'|\, dt$$

$\|$

$|v|$

which is impossible

② $\gamma \subset B(\bar{r})$



$$\int_0^1 |\gamma'|\, dt \geq r(\gamma(1)) = |v|$$

$\|$

$L(\gamma) = |v|$

$\Rightarrow$ Inequality is in fact equality

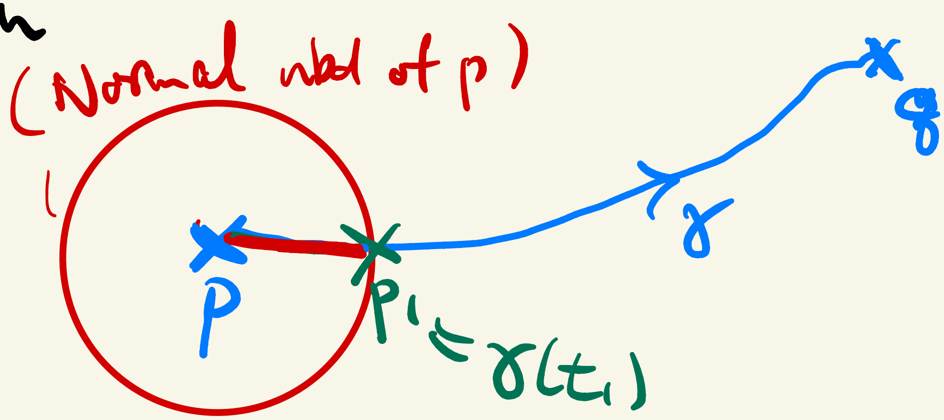
$$\Rightarrow \quad \gamma'(t) = \lambda(t)\,\nabla r = \lambda(t)\,\frac{\partial}{\partial r}.$$

for some $\lambda(t) \geq 0$.

$\therefore$ $\gamma(t) = \gamma_v(t)$ after Reparametrization as a curve.

For ② pf: If $L(\gamma) = d(\gamma(0), \gamma(1))$

then



① $\Rightarrow$ $\gamma|_{[0,t_1]}$ = geodesic otherwise

$\exists\ \tilde{\gamma}|_{[0,t_1]}$ s.t. $L(\tilde{\gamma}) < L(\gamma|_{[0,t_1]})$

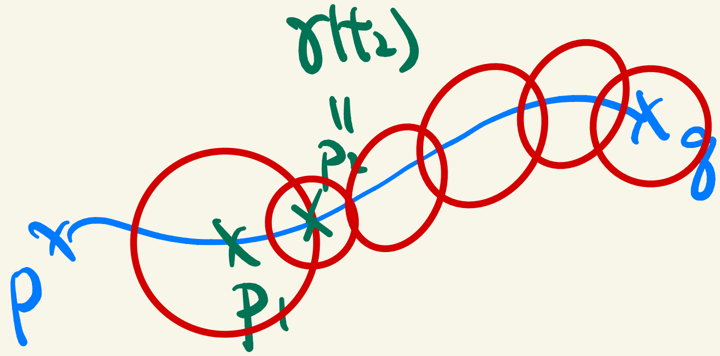
$\Rightarrow$ $\tilde{\gamma}|_{[0,t_1]} \cup \gamma|_{[t_1,1]}$ = curve from p to q with shorter length

Which is impossible.

Unless: t_1 doesn't exist.

$\Rightarrow \gamma \subseteq B(\bar{r}) \overset{①}{\Rightarrow} \gamma = $ smooth geod.

Inductively,



By local unique

$\Downarrow$

either $\exists p_2$ s.t. $\gamma|_{[0,t_2]}$ = geod. (smooth)

OR $\gamma|_{[t_1,1]} \subseteq$ Normal nbd of p_1.

$\vdots$

The process must be terminated at k-th step.

$\divideontimes$

Hopf - Rinow thm : The following are equ.:

① (M, d) is a complete metric space.

② $\exists\, p \in M$, s.t. $\exp_p : T_pM \to M$ is well-def.

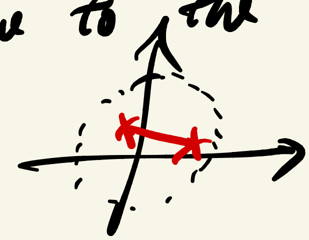
③ $\forall\, p \in M$, $\exp_p : T_pM \to M$ is well-defined.

$\Downarrow$

④ $\forall\, p, q \in M$, $\exists\, \gamma$ geod. s.t. $L(\gamma) = d(p, q)$.

Rmk: (i) ① holds automatically if M = cpt.

(ii) ④ $\Rightarrow$ ① (⇔ ② ⇔ ③) due to the counter-example of disk.



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pf: ③ ⇒ ② ✓

① ⇒ ③: Fix $p \in M$, $v \in T_pM$.

Claim: γ_v is well-defined up to $t = 1$.

pf of claim:

let $t_1 > 0$ be s.t. γ_v exists up to t_1

$t_1 = \sup\{t>0 \mid \gamma_v \text{ exists on } [0,t)\}$

where $\gamma_v(0) = p$, $\gamma_v'(0) = v$. and $\nabla_{\gamma_v'}\gamma_v' = 0$.

By ODE, $t_1 > 0$.

- Suffices to show that $\gamma_v(t_1)$ exists unless $t_1 = \infty$

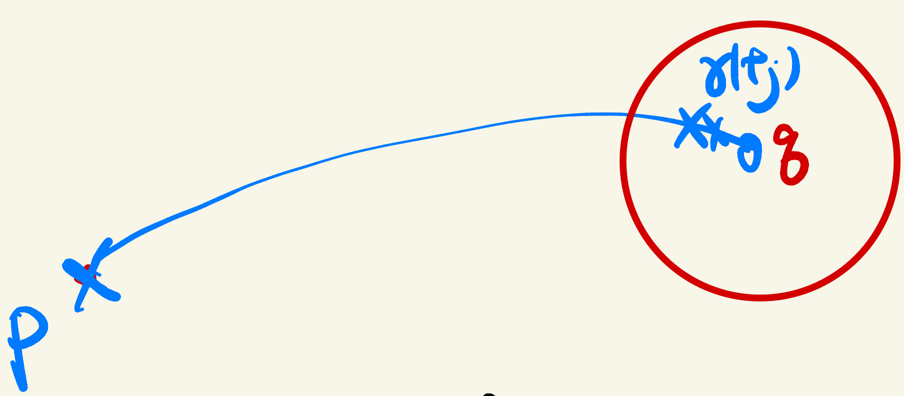
  (since then ODE existence $\Rightarrow$ γ_v can be extended across t_1 which is impossible.)

Let $t_j \uparrow t_1$ (if $t_1 < \infty$)

$\gamma(t_j) \in M \Rightarrow d(\gamma(t_j), \gamma(t_i)) = |v| \cdot |t_i - t_j|$

$\therefore \{\gamma(t_j)\}$ is a Cauchy seq in M

$\Rightarrow \exists\, q \in M$ s.t. $\gamma(t_j) \xrightarrow{j\to\infty} q \in M$.



$\therefore \gamma(t_j) \subset$ Normal nbd of q $\forall j \gg 1$.

$\therefore \exists!$ smooth minimizing geo from $\gamma(t_j)$ to q inside normal nbd of q

γ thanks to uniqueness!!

$\Rightarrow$ γ can be extended to q and hence on $[0, t_1+\varepsilon]$ for some $\varepsilon>0$, which is smooth.

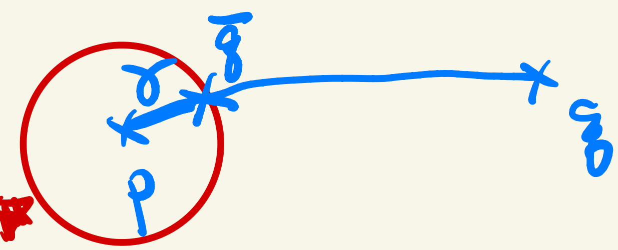
$\therefore$ t_1 can't be maximal, if finite $\Rightarrow t_1 = +\infty$ #.

---

pf of ② $\Rightarrow$ ① ($\exp_p$ defined globally $\Rightarrow$ complete)

Claim: $\forall p, q \in M$, $\exists \gamma : p \to q$ which is a geodesic s.t. $d(p,q) = L(\gamma)$ ($\gamma : [0,1] \to M$) ④

pf:



Normal nbd at p (shrink it if needed)

(If $q \in$ normal nbd of p, then trivial by previous Result.)

$\exists\, \bar{q} \in \partial B(\bar{r})$ s.t. $d(p,q) = d(p,\bar{q}) + d(\bar{q},q)$.

by cptness of $\partial B(\bar{r})$. $(\bar{r} << 1)$

Realized by geodesic!!

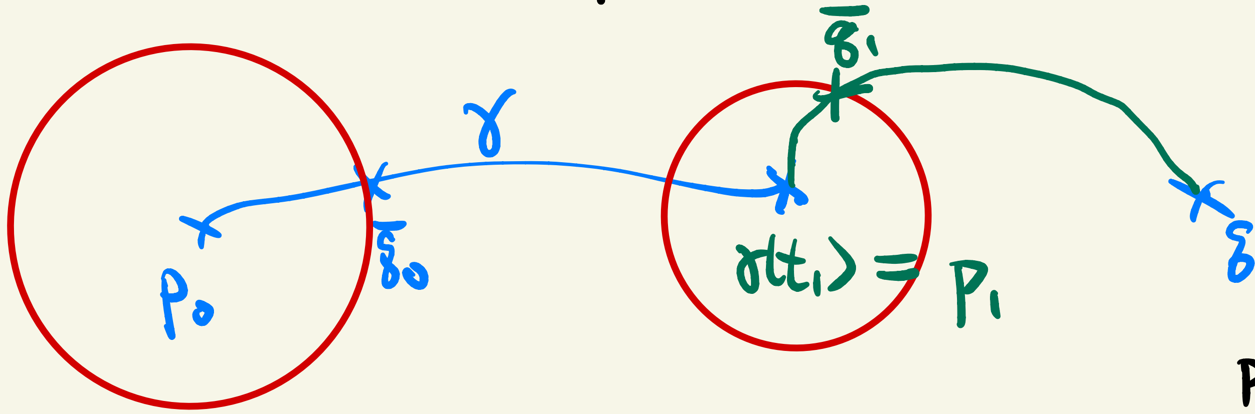
Let γ be the Normal ($|\gamma'|=1$) geodesic from p to $\bar{q}$.

Let $I = \{ t \geq 0 : d(p,\gamma(t)) + d(\gamma(t),q) = d(p,q) \}$

$\Rightarrow I \neq \phi$ by our choice.

claim: $I = [0, d(p,q)]$.

pf: suppose $t_1 = \sup I < d(p,q)$.



Same arguement, $\exists\, \bar{q}_1 \in$ Normal nbd of $\gamma(t_1)$ ($= p_1$)

s.t. $d(\gamma(t_1), q) = d(\gamma(t_1), \bar{q}_1) + d(\bar{q}_1, q)$

$\Downarrow$

$$d(p,q) = d(p,\gamma(t_1)) + d(\gamma(t_1), q) \quad (\text{def of } t_1)$$

$$= d(p,\gamma(t_1)) + d(\gamma(t_1),\bar{q}_1) + d(\bar{q}_1, q)$$

$$\Rightarrow d(p,\gamma(t_1)) + \boxed{d(\gamma(t_1),\bar{q}_1)} = d(p,\bar{q}_1)$$

$$\|$$

$$d(\tilde{\gamma}(t_1),\tilde{\gamma}(t_2))$$

for some $\tilde{\gamma} : [t_1, t_2] \to M$ s.t.

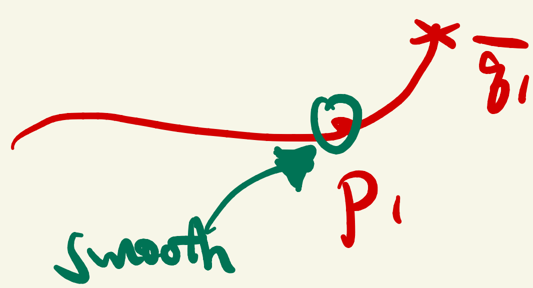
$\tilde{\gamma}$ = smooth geo. from $\gamma(t_1) = p_1$ to $\bar{q}_1$.

$\therefore$ If we extend γ s.t. $\gamma(t) = \tilde{\gamma}(t)$ for $t \in [t_1, t_2]$.

then we have

$$d(\gamma(0), \gamma(t_2)) = d(p, \bar{q}_1) = L(\gamma)$$

Earlier thm $\Rightarrow$ γ = smooth geodesic



(original γ is extensible smoothly)

And hence

$$d(p,q) = d(p,\gamma(t_1)) + d(\gamma(t_1),q)$$
$$= d(p,\gamma(t_1)) + d(\gamma(t_1),\gamma(t_2)) + d(\gamma(t_2),q)$$

γ = smooth

$$= d(p,\gamma(t_2)) + d(\gamma(t_2),q)$$

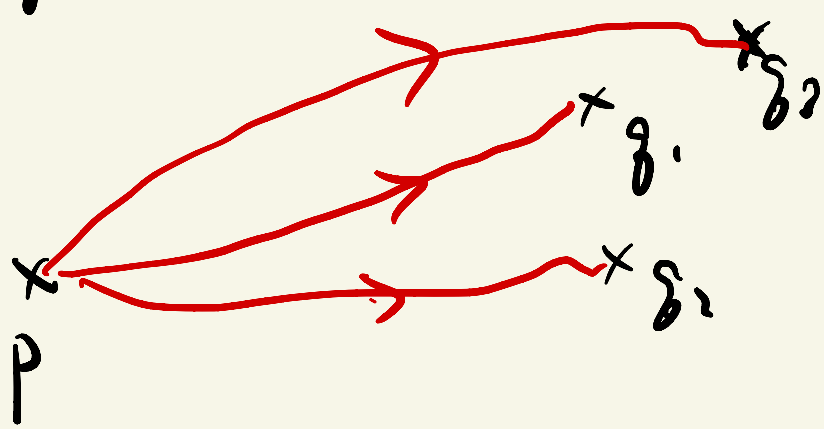
where $t_2 > t_1$ $\therefore$ impossible!!

$$\therefore I = [0, d(p,q)] \rightsquigarrow \gamma(d(p,q)) = q.$$

This proved ④.

Pf of ①: ((M, d) is complete)

Let $\{q_i\}$ be Cauchy in M.



$\exists$ Normal good. $\gamma_i : [0, t_i] \to M$ s.t.

$\gamma_i(0) = p$, $\gamma_i(t_i) = q_i$. } By ④.

and $|\gamma_i'(0)| = 1$

Note: $t_i = d(p, q_i)$ which is bdd

since $|d(p, q_i) - d(p, q_j)| \leq d(q_i, q_j)$

by tri. ineq. $\forall i, j$.

$\Rightarrow$ $\{t_i\}$ is cauchy

$\Rightarrow$ $t_i \to t_\infty \in [0, +\infty)$

case 1: $t_\infty = 0 \Rightarrow d(p, q_i) \to 0$ #

case 2: $t_\infty > 0$.

$\gamma_i'(0) = v_i \in S^{n-1} \subseteq T_pM.$

$\therefore \exists\, v_\infty$ s.t. after passing to sub.

$v_{i_j} \to v_\infty$ as $j \to \infty$.

Stability of ODE

$\Rightarrow \gamma_{i_j} \to \gamma_\infty$ where

γ_∞ is a geodesic from p with

$\gamma_\infty'(0) = v_\infty$.

Because $\{t_{i_j}\}$ is bdd.

$$\Rightarrow \quad \gamma_\infty(t_\infty) = \lim_{j\to\infty} \gamma_{i_j}(t_{i_j})$$

$$= \lim_{j\to\infty} q_{i_j}$$

$$\Rightarrow \quad q_i \to q_\infty \triangleq \gamma_\infty(t_\infty) \quad \#.$$