Tutorial 7

· A continuous but nowhere differentiable function.

for $0<\alpha<1$, $f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$

Recall in Ex 3.3.15, actually we have proved it to be C^α Hölder continuous.

Sketch of the proof: $\Delta_N(g) = 2\sigma_{2N}(g) - \sigma_N(g)$ delayed means

- If g cont. & diff. at x_0, then $\Delta_N(g')(x_0) = O(\log N)$.
- $\Delta_{2N}(f) - \Delta_N(f) = 2^{-n\alpha} e^{i2^n x}$

4.5.9 Prove that $\langle a \log n \rangle$ is not equidistributed for any a.

Proof: The case $a=0$ is trivial. For $a\neq 0$, it follows the Weyl's criterion that

it suffices to prove that there exist $k\neq 0$ s.t.

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a \log n} \to 0 \text{ as } N\to\infty$$

does not hold.

For $b\neq 0$,

$$\left|\sum_{n=1}^{N}\left(e^{2\pi i b \log n}\right) - \int_1^N e^{2\pi i b \log x}\,dx\right|$$

$$\leq \sum_{n=1}^{N-1}\int_n^{n+1}\left|e^{2\pi i b\log(n+1)} - e^{2\pi i b \log x}\right| dx + 1$$

$$\leq 2\pi|b|\sum_{n=1}^{N-1}\int_n^{n+1}\left|\log(n+1) - \log x\right| dx + 1$$

$$\leq 2\pi|b| \log N + 1 = O(\log N)$$

$$\int_1^N e^{2\pi i b \log x}\, dx = \int_1^N x^{2\pi i b}\, dx = \frac{1}{2\pi i b}\left(N^{2\pi i b+1}-1\right).$$

Therefore, $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b \log n} - \frac{1}{2\pi i b} N^{2\pi i b} \to 0$ as $N\to\infty$.

But $\frac{1}{2\pi i b} N^{2\pi i b}$ is oscillating as $N\to\infty$, and thus

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b \log n} \to 0 \text{ does not hold for any } a \neq 0.$$

$\square$

<u>4.6.4</u> ("pilling up singularities") On $[-1,1)$, consider

$$\varphi(x) = |x|.$$

and exend φ to $\mathbb{R}$ and periodic of period 2. Define f as

$$f(x) = \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \varphi(4^n x)$$

It is obviously cont. on $\mathbb{R}$.

(a) Fix $x_0 \in \mathbb{R}$. For any $m \in \mathbb{N}^+$. $\delta_m = \pm \frac{1}{2} 4^{-m}$ where the sign is chosen such that no integer lies in between $4^m x_0$ and $4^m(x_0+\delta_m)$. Consider the quotient

$$\gamma_n = \frac{\varphi(4^n(x_0+\delta_m)) - \varphi(4^n x_0)}{\delta_m}$$

Prove that if $n>m$, then $\gamma_n = 0$ & for $0 \le n \le m$ one has $|\gamma_n| = 4^n$

(b) Prove that $\left|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\right| \ge \frac{1}{2}(3^m+1)$, and conclude f is not diff at x_0

Proof. (a) If $n > m$, $|4^n \delta_m| = 2\,4^{n-m-1}$, which is the multiple of period 2. Therefore,

$$\gamma_n = \frac{\varphi(4^n(x_0+\delta_m)) - \varphi(4^n(x_0))}{\delta_m} = 0;$$

If $0 \le n \le m$, ~~since no integer lies in between $4^m x_0$ and $4^m(x_0+\delta_m)$, neither in between $4^n x_0$ and $4^n(x_0+\delta_m)$.~~ no integer lies between $4^n x_0$ and $4^n(x_0+\delta_m)$, otherwise that integer multiplied by 4^{n-m} will lie between $4^m x_0$ and $4^m(x_0+\delta_m)$. Therefore it follows from the Mean Value Theorem that

$$|\gamma_n| = \frac{|4^n \delta_m|}{|\delta_m|} = 4^n.$$

(b)

$$\left|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \frac{\varphi(4^n(x_0+\delta_m))-\varphi(4^n x_0)}{\delta_m}\right|$$

$$= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right| \ge \left(\frac{3}{4}\right)^m |\gamma_m| - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n |\gamma_n|$$

$$= 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m+1)$$

Therefore, as $m \to \infty$ one has $\delta_m \to 0$, and

$$\left|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\right| \to \infty.$$

which implies that f is not diff. at x_0 $\square$