Tutorial 6

Γ is a simple closed curve & parametrized by $\gamma(s) = (x(s), y(s))$

length $\ell = \int_a^b |\gamma'(s)| ds = \int_a^b (x'(s)^2 + y'(s)^2)^{1/2} ds$

area $A = \frac{1}{2} \left| \int_a^b x(s) y'(s) - x'(s) y(s) \, ds \right|$.

<u>isoperimetric ineq.</u> $A \leq \frac{\ell^2}{4\pi}$

the equality holds iff Γ is a circle.

$\{\xi_n\}_{n=1}^{\infty} \subset [0,1)$ is said to be equidistributed if

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N ; \xi_n \in (a,b)\}}{N} = b-a.$$

<u>Thm 2.1</u> If γ is irrational, then $\langle n\gamma \rangle$ is equidistributed in $[0,1)$

<u>Weyl's Criterion</u> $\xi_1, \xi_2, \cdots$ equidistributed in $[0,1)$

$$\Longleftrightarrow \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0 \quad \text{as } N \to \infty.$$

$$\left(\begin{array}{l} \sim \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b)}(n\gamma) \to \int_0^1 \chi_{[a,b)}(x)\, dx \\ \sim \text{any cont. \& periodic of period 1 function} \\ \sim \text{trigonometric polynomials} \end{array} \right)$$

4.5.1 Prove that any curve Γ admits a parametrization by arc-length.

Proof. Suppose η is any parametrization, and $h(s) = \int_a^s |\eta'(t)|\,dt$.

Then $h(s)$ is the arc-length. Since h is monotonely increasing, the inverse of h exists and $(h^{-1})'(t) = \frac{1}{h'(h^{-1}(t))}$.

Therefore

$$\left|(\eta \circ h^{-1})'(t)\right| = \left|\eta'(h^{-1}(t))\right| \frac{1}{\left|h'(h^{-1}(t))\right|}$$

$$= \left|\eta'(h^{-1}(t))\right| \frac{1}{\left|\eta'(h^{-1}(t))\right|} = 1.$$

□

4.5.5. Prove that $\langle (\frac{1+\sqrt{5}}{2})^n \rangle$ is not distributed in $[0,1)$.

Proof. $U_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ is the solution of the difference equation $U_{r+1} = U_r + U_{r-1}$ with $U_1 = 1$, $U_2 = 3$. So U_n are integers.

Note that $\left(\frac{1-\sqrt{5}}{2}\right) \in (-1, 0)$. Therefore, when n is large $\left(\frac{1-\sqrt{5}}{2}\right)^n \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, and then the fractional part of $\left(\frac{1+\sqrt{5}}{2}\right)^n$ is in $[0, \varepsilon_0) \cup (1-\varepsilon_0, 1)$.

So $\langle (\frac{1+\sqrt{5}}{2})^n \rangle$ is not equidistributed in $[0,1)$.

□

$\underline{4.5.8}$ Show that for any $a \neq 0$, σ satisfying $0<\sigma<1$, the sequence $\langle an^\sigma \rangle$ is equidistributed in $[0,1)$.

(Hint: Prove that $\sum_{n=1}^{N} e^{2\pi i b n^\sigma} = O(N^\sigma) + O(N^{1-\sigma})$ if $b \neq 0$.)

$\underline{\text{Proof.}}$ $\langle an^\sigma \rangle$ is equidistributed in $[0,1)$

$$\Leftrightarrow \quad \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(an^\sigma) \to \int_0^1 \chi_{[a,b)}(x)\,dx \quad \text{as } N \to \infty.$$

Since $\chi_{[a,b)}$ can be approximated by trigonometric polynomials, it suffices to prove $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a n^\sigma} \to 0$ as $N \to \infty$ for $k \in \mathbb{N}$, $k \neq 0$.

Let $b \triangleq ak$:

$$\left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| = \left| \frac{x^{1-\sigma}}{2\pi i b\sigma} e^{2\pi i b x^\sigma}\Big|_1^N - \int_1^N \frac{(1-\sigma)x^{-\sigma}}{2\pi i b\sigma} e^{2\pi i b x^\sigma}\,dx \right|$$

$$= \left| \frac{N^{1-\sigma} e^{2\pi i b N^\sigma} - e^{2\pi i b}}{2\pi i b\sigma} \right| + \int_1^N \frac{(1-\sigma)x^{-\sigma}}{|2\pi i b\sigma|}\,dx$$

$$\leq \frac{N^{1-\sigma}+1}{2\pi |b|\sigma} + \frac{(1-\sigma)}{2\pi|b|\sigma}\int_1^N x^{-\sigma}\,dx = O(N^{1-\sigma})$$

$$\left|\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \int_1^N e^{2\pi i b x^\sigma}\,dx\right| = \left|\sum_{n=1}^{N-1}\int_n^{n+1}\left(e^{2\pi i b n^\sigma} - e^{2\pi i b x^\sigma}\right)dx + e^{2\pi i b N^\sigma}\right|$$

$$\leq \sum_{n=1}^{N-1}\int_n^{n+1} \left|e^{2\pi i b n^\sigma} - e^{2\pi i b x^\sigma}\right| dx + 1$$

$$\leq \sum_{n=1}^{N-1}\int_n^{n+1} 2\pi b\,|x^\sigma - n^\sigma|\,dx + 1$$

$$\leq 2\pi b \sum_{n=1}^{N-1}\int_n^{n+1} |(n+1)^\sigma - n^\sigma|\,dx + 1$$

$$= 2\pi b\,(N^\sigma - 1) + 1 = O(N^\sigma).$$

So $\sum_{n=1}^{N} e^{2\pi i b n^\sigma} = O(N^\sigma) + O(N^{1-\sigma})$, and then $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i b n^\sigma} \to 0$. $\square$