Tutorial 5

2.3.11 (a) If f is T-periodic, continuous, piecewise C^1 with $\int_0^T f(t)\,dt = 0$

Show that
$$\int_0^T |f(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2 dt$$
with ineq. if and only if
$$f(t) = A\sin\left(\frac{2\pi}{T}t\right) + B\cos\left(\frac{2\pi}{T}t\right).$$

(b) If f is as above & $g \in C^1$, T-periodic, prove that
$$\left|\int_0^T \overline{f(t)}\, g(t)\,dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2 dt \int_0^T |g'(t)|^2 dt$$

(c) For any compact interval $[a,b]$ & any continuously differentiable function f with $f(a)=f(b)=0$, show that
$$\int_a^b |f(t)|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2 dt.$$
Show that the constant $\frac{(b-a)^2}{\pi^2}$ can not be improved.

Proof. (a) $\hat{f}(n) = \frac{1}{T}\int_0^T f(t)\, e^{-in\frac{2\pi}{T}t}\,dt$. $\hat{f}(0) = \frac{1}{T}\int_0^T f(t)\,dt = 0$.

$$\widehat{f'}(n) = \frac{1}{T}\int_0^T f'(t)\, e^{-in\frac{2\pi}{T}t}\,dt$$
$$= -\frac{1}{T}\int_0^T f(t)\left(-in\frac{2\pi}{T}\right) e^{-in\frac{2\pi}{T}t}\,dt$$
$$= in\frac{2\pi}{T}\hat{f}(n).$$

Therefore, $|\widehat{f'}(n)| \ge \frac{2\pi}{T}|\hat{f}(n)|$, for $n \in \mathbb{Z}$

$$\int_0^T |f(t)|^2 dt = T\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \le \frac{T^2}{4\pi^2}\, T\sum_{n\in\mathbb{Z}} |\widehat{f'}(n)|^2$$
$$= \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2 dt$$

Since if $\hat{f}(n) \neq 0$, then $\left|\widehat{f'}(n)\right| > \frac{2\pi}{T}\left|\hat{f}(n)\right|$ for $n \neq 0, \pm 1$,

one has that the above inequality holds if and only if

$\hat{f}(n) = 0$ for $n \neq 0, \pm 1$, i.e.

$$f(t) = A \sin\left(\frac{2\pi t}{T}\right) + B \cos\left(\frac{2\pi t}{T}\right).$$

(b) It follows from Lemma 1.5 & Cauchy-Schwartz ineq. that

$$\left| \int_0^T \overline{f(t)}\, g(t)\, dt \right|^2 = T^2 \left| \sum_{n=-\infty}^{\infty} \overline{\hat{f}(n)}\, \hat{g}(n) \right|^2 = T^2 \left| \sum_{n\neq 0} \overline{\hat{f}(n)}\, \hat{g}(n) \right|^2$$

$$\leq T^2 \left(\sum_{n\neq 0} |\hat{f}(n)|^2 \right) \left(\sum_{n\neq 0} |\hat{g}(n)|^2 \right)$$

$$\leq T^2 \left(\sum_{n\neq 0} |\hat{f}(n)|^2 \right) \frac{T^2}{4\pi^2} \left(\sum_{n\neq 0} |\widehat{g'}(n)|^2 \right)$$

$$= \frac{T^2}{4\pi^2} \int_0^T |f(t)|^2\, dt \int_0^T |g'(t)|^2\, dt$$

(c) Extend f to be odd with respect to a on $[2a-b, b]$, and

periodic of $T = 2(b-a)$. Then $\int_{2a-b}^{b} f(t)\, dt = 0$.

So it follows from the result of (a) that

$$\int_a^b |f(t)|^2\, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|\, dt.$$

The equality holds if $f(t) = A \sin\left(\pi \frac{t-a}{b-a}\right) + B \cos\left(\pi \frac{t-a}{b-a}\right)$.

Since we suppose that f is odd with respect to a $\square$

then $f(t) = A \sin\left(\pi \frac{t-a}{b-a}\right)$.

3.3.12 Prove that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Proof. Consider the Dirichlet kernel, $D_N(x) = \sum_{-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$.

$$\int_{-\pi}^{\pi} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}\,dx = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{inx}\,dx = 2\pi.$$

~~Note that~~ $\cancel{\int_{\frac{\pi}{2}}^{\pi} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}\,dx =}$

Then
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin((2N+1)x)}{x}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin((2N+1)x)}{\sin x}\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{x} - \frac{1}{\sin x}\right)\sin((2N+1)x)\,dx$$
$$= \pi + \int_{-\pi}^{\pi} \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}\left(\frac{1}{x} - \frac{1}{\sin x}\right)\sin((2N+1)x)\,dx.$$

Since $\frac{1}{x} - \frac{1}{\sin x}$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$ is a Riemann integral function on $[-\pi, \pi)$.

Therefore, Riemann-Lebesgue lemma implies that as $N \to \infty$,
$$\int_{-\pi}^{\pi} \chi_{[-\frac{\pi}{2},\frac{\pi}{2})}\left(\frac{1}{x} - \frac{1}{\sin x}\right)\sin((2N+1)x)\,dx \to 0$$

Noting that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin((2N+1)x)}{x}\,dx = \int_{-(2N+1)\frac{\pi}{2}}^{(2N+1)\frac{\pi}{2}} \frac{\sin x}{x}\,dx$, one has

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi, \text{ i.e. } \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

$\square$

3.3 15. Let f be a 2π-periodic & Riemann integrable function on $[-\pi, \pi]$.

f satisfies a Hölder condition of order α, i.e.

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for some $0 < \alpha \leq 1$, $C > 0$ and all x, h. Show that

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$

Proof $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\left(-e^{-in(x-\frac{\pi}{n})}\right)dx\right)$

$$= -\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n}) e^{-inx}\,dx.$$

Therefore,

$$|\hat{f}(n)| = \left|\frac{1}{4\pi}\int_{-\pi}^{\pi}\left[f(x) - f(x+\tfrac{\pi}{n})\right] e^{-inx}\,dx\right|$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\tfrac{\pi}{n})|\,dx$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{C}{|n|^{\alpha}}\,dx \leq \frac{C}{2|n|^{\alpha}}.$$

□

Smoothness of f $\Rightarrow$ decay of $\hat{f}(n)$