Tutorial 4

- vector space, inner product, Hilbert space.

↳ $\mathcal{R}$ (the set of complex-valued Riemann integrable fcns on $[0, 2\pi]$)

is not complete under the norm

$$\|f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta\right)^{1/2}$$

- Mean-square convergence

① f integrable, ② $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)|^2 d\theta \to 0$$

↳ Parseval's identity $\quad \sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta$

Bessel's inequality $\quad \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2$

2.7.2. $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} = \dfrac{\sin\left((N+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$. $\quad L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta$

Prove that

$$L_N \geq c\log N, \quad \text{for some } c > 0.$$

Remark: It shows that the Dirichlet kernel is not a good kernel.

Proof. $\theta \in [-\pi, \pi]$, then $\frac{\theta}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Therefore, $|\sin\frac{\theta}{2}| \leq \frac{|\theta|}{2}$, and $|D_N(\theta)| \geq \dfrac{2|\sin\left((N+\frac{1}{2})\theta\right)|}{|\theta|}$

$$L_N \geq \frac{4}{2\pi}\int_0^{\pi} \frac{|\sin((N+\frac{1}{2})\theta)|}{|\theta|} d\theta = \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin\theta|}{|\theta|} d\theta \geq \frac{2}{\pi}\int_{\pi}^{N\pi} \frac{|\sin\theta|}{|\theta|} d\theta$$

$$= \sum_{k=1}^{N-1} \frac{2}{\pi} \int_{k\pi}^{(k+1)\pi} \frac{|\sin\theta|}{|\theta|} d\theta \geq \sum_{k=1}^{N-1} \frac{2\int_0^{\pi} |\sin\theta| d\theta}{(k+1)\pi^2} \geq \sum_{k=2}^{N} \frac{4}{k\pi^2}$$

$$\geq \frac{2}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \geq C \log N.$$

$\square$

33 5. $f(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ \log(1/\theta) & \text{for } \theta \in (0, 2\pi] \end{cases}$

$f_n(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \frac{1}{n} \\ f(\theta) & \text{for } \frac{1}{n} \leq \theta < 2\pi \end{cases}$

Prove that $\{f_n(\theta)\}_{n=1}^{\infty}$ is a Cauchy sequence in R.

However, f does not belong to R.

Proof. $|f_n(\theta) - f_m(\theta)| = \begin{cases} 0 & \text{otherwise.} \\ f(\theta) & \text{for } \frac{1}{m} < \theta < \frac{1}{n} \end{cases}$ for any $n < m$

Therefore, $\| f_n(\theta) - f_m(\theta) \|^2 = \frac{1}{2\pi} \int_{\frac{1}{m}}^{\frac{1}{n}} \left(\log\left(\frac{1}{\theta}\right)\right)^2 d\theta$

$$\leq \frac{1}{2\pi} \int_n^m \frac{(\log x)^2}{x^2} dx \leq \frac{1}{2\pi} \int_n^m \frac{(\log x)^2}{x^{1/2}} \frac{1}{x^{\frac{3}{2}}} dx$$

For any $\varepsilon > 0$, $\exists N \in \mathbb{N}^*$. s.t. for any $x > N$

$$0 < \frac{(\log x)^2}{x^{\frac{1}{2}}} < \pi\varepsilon.$$

Hence, for any $n > \max\{N, \frac{1}{\varepsilon^2}\}$

$$\| f_n(\theta) - f_m(\theta) \| \leq \left(\frac{1}{2\pi} \pi\varepsilon \cdot 2x^{-\frac{1}{2}} \Big|_m^n \right)^{1/2} < \varepsilon.$$

However, since f is not bounded, $f \notin R$. $\square$

3.2.7 Show that

$$\sum_{n\geq 2} \frac{1}{\log n} \sin nx$$

converges ~~forp~~ for every x, yet it is not the Fourier series of a Riemann integrable function.

Proof. Since $\frac{1}{\log n}$ is monotonely decreasing for $n \geq 2$.

$$\frac{1}{\log n} \to 0 \text{ as } n \to \infty, \quad \tilde{D}_N(x) \triangleq \sum_{n=2}^{N} \sin nx = \begin{cases} 0 & \text{for } x = 0, 2\pi \\ \dfrac{\cos\left(\frac{3}{2}x\right) - \cos\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}} & \text{for } x \in (0, 2\pi) \end{cases}$$

then it follows from the Dirichlet's test for convergence that

$$\sum_{n\geq 2} \frac{1}{\log n} \sin nx \text{ converges for every } x.$$

Suppose that it were the Fourier series of a Riemann integrable function $f(x)$. Then by Parseval's identity,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty.$$

However,
$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0, \pm 1}} \left| \frac{\operatorname{sgn}(n)}{(2i)\log n} \right|^2 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{(\log n)^2}$$

$$\geq c \sum_{n=2}^{\infty} \frac{1}{n} = \infty \quad \text{for some } c > 0.$$

Since $\lim_{n\to\infty} \frac{\log n}{n^{1/2}} = 0$. □