Tutorial 2

The following were discussed in the tutorial this week:

1. Let f be a Riemann integrable function on $[-\pi, \pi]$. Then the Fourier series of f is given by
$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},$$
where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy$. Equivalently
$$f \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$
where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\, dy$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\, dy$ for $n \geq 1$. Find the relation between $\hat{f}(n)$ and a_n, b_n.

2. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} -\pi/2 - x/2 & \text{if } -\pi \leq x < 0, \\ 0 & \text{if } x = 0, \\ \pi/2 - x/2 & \text{if } 0 < x \leq \pi. \end{cases}$$
   (a) Compute the Fourier coefficients $\hat{f}(n)$, a_n and b_n of f. Verify that they do satisfy the relation obtained in Q1.

   (b) Show that the Fourier series of f converges at every point $x \in [-\pi, \pi]$. (**Hint:** you may apply the Dirichlet's test.)

3. We say that the Fourier series of f converges absolutely if
$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$
Let $-\pi < a < b < \pi$. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the characteristic function of $[a, b]$, that is
$$f(x) = \chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Find the Fourier series of f.

   (b) Show that the Fourier series of f converges at every $x \in [-\pi, \pi]$.

   (c) Show that the Fourier series of f is NOT absolutely convergent. (**Hint:** Let $\theta_0 = (b - a)/2$. Note $\theta_0 \in (0, \pi)$. Find an interval $[c, \pi - c] \subset (0, \pi)$ that has length $> \theta_0$. Show that for any $k \geq 1$, there is $n_k \geq 1$ such that $k\pi + [c, \pi - c]$ contains $n_k\theta_0$.)

1. Sol. If $n \geq 1$. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \cos ny \, dy$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \frac{e^{iny}+e^{-iny}}{2} dy$$

$$= \hat{f}(n) + \hat{f}(-n)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \sin ny \, dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \frac{e^{iny}-e^{-iny}}{2i} dy$$

$$= i\left(\hat{f}(n) - \hat{f}(-n)\right)$$

if $n=0$ $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, dy = \hat{f}(0)$

$\square$

2. Sol. (a) $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iny} dy = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(-\frac{\pi}{2}-\frac{y}{2}\right)e^{-iny}dy + \int_{0}^{\pi}\left(\frac{\pi}{2}-\frac{y}{2}\right)e^{-iny}dy\right]$

$$= \frac{1}{2\pi}\left[\left(-\frac{\pi}{2}-\frac{y}{2}\right)\frac{1}{-in}e^{-iny}\Big|_{-\pi}^{0} - \int_{0}^{\pi}\left(-\frac{1}{2}\right)\frac{e^{-iny}}{-in}dy\right.$$

$$\left.+\left(\frac{\pi}{2}-\frac{y}{2}\right)\frac{1}{-in}e^{-iny}\Big|_{0}^{\pi} - \int_{0}^{\pi}\left(-\frac{1}{2}\right)\frac{e^{-iny}}{-in}dy\right]$$

$$= \frac{1}{2\pi}\left(-\frac{\pi}{n}i\right) = -\frac{i}{2n}, \quad \text{for } n \in \mathbb{N}.$$

$a_n = 0$ for $n \geq 0$, $n \in \mathbb{N}$, since $f(y)$ is an odd function.

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(-\frac{\pi}{2}-\frac{y}{2}\right)\sin ny\, dy + \int_{0}^{\pi}\left(\frac{\pi}{2}-\frac{y}{2}\right)\sin ny\, dy\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}-\frac{y}{2}\right)\frac{\cos ny}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi} -\frac{1}{2}\frac{\cos ny}{n}dy\right]$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2n}\right) = -\frac{1}{n} \quad \text{for } n \geq 1.\ n \in \mathbb{N}.$$

(b) $f \sim \sum_{n=1}^{\infty} -\frac{1}{n} \sin nx$

When $x = 0, \pm\pi$. the series obviously converges;

When $x \neq 0$ or $\pm\pi$,

$$\left| \sum_{i=1}^{N} \sin n\pi \right| = \frac{1}{|\sin x|} \left| \sum_{i=1}^{N} \sin x \sin nx \right|$$

$$= \frac{1}{|\sin x|} \left| \sum_{i=1}^{N} \frac{\cos(n-1)x - \cos(n+1)x}{2} \right|$$

$$= \frac{1}{|\sin x|} \frac{|1+\cos x - \cos Nx - \cos(N+1)x|}{2} \leq \frac{2}{|\sin x|}$$

$\frac{1}{n+1} \leq \frac{1}{n}$. $\lim_{n\to\infty} \frac{1}{n} = 0$.

and then it follows from the Dirichlet's test that the series converges for all $x \in [-\pi, \pi]$.

□

3. Sol. (a) $\hat{f}(n) = \frac{1}{2\pi} \int_a^b e^{-iny}\, dy = \frac{i}{2n\pi}\left(e^{-inb} - e^{-ina}\right)$

So the Fourier series of f is $\sum_{n=-\infty}^{\infty} \frac{i}{2n\pi}\left(e^{-inb} - e^{-ina}\right) e^{inx}$

(b) Considering $\sum_{n=-N}^{N} \frac{i}{2n\pi}\left(e^{-inb} - e^{-ina}\right) e^{inx}$

$$= \sum_{n=1}^{N} \frac{i}{2n\pi}\left(e^{-inb} e^{inx} - e^{-ina} e^{inx} - e^{inb} e^{-inx} + e^{ina} e^{-inx}\right)$$

$$= \sum_{n=1}^{N} \frac{1}{n\pi}\left(\sin n(b-x) - \sin n(a-x)\right)$$

Similar with 2(b). so we omit the proof here.

(c) $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{2n\pi} \left| \cos nb - \cos na - i(\sin nb - \sin na) \right|$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2n\pi} \left| -2\sin n\frac{b+a}{2} \sin n\frac{b-a}{2} - 2i \cos n\frac{b+a}{2} \sin n\frac{b-a}{2} \right|$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{|n|\pi} \left| \sin n\frac{b-a}{2} \right| .$$

Since $-\pi < a < b < \pi$, we have that

there exists an interval $[c, \pi - c] \subset (0, \pi)$ such that

$$\pi - 2c > \frac{b-a}{2} \triangleq \theta_0 .$$

Define $\varepsilon = \sin c$. Let n_k satisfy

$$2k\pi + c < n_k \theta_0 < 2k\pi + \pi - c ,$$

$$(\Longleftrightarrow) \quad \frac{2k\pi + c}{\theta_0} < n_k < \frac{2k\pi + \pi - c}{\theta_0} .$$

Note that $\frac{\pi - 2c}{\theta_0} > 1$.

Therefore, for any $k \geq 1$, $k \in \mathbb{N}$, $\exists\, n_k \in \mathbb{N}$

s.t. $\sin n_k \theta_0 > \varepsilon$ and $\frac{1}{n_k} > \frac{\theta_0}{2k\pi + \pi - c}$.

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{|n|\pi} |\sin n\theta_0|$$

$$> \sum_{k=1}^{\infty} \frac{\theta_0}{2k\pi + \pi - c} \varepsilon > \sum_{k=1}^{\infty} \frac{\theta_0 \varepsilon}{4\pi} \frac{1}{k} = \infty .$$

So the Fourier series of f is NOT absolutely convergent.

$\square$