Tutorial 1

① $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ wave eqn.

$$\begin{cases} \cancel{u(0,t) = u(\pi,t) = 0, \quad t>0} \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

② $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ heat eqn.

Steady-state heat eqn in the disc.

$$\Delta u = 0 \implies r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}$$

Suppose that $u(r,\theta) = F(r)\,G(\theta)$

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}$$

$$\implies \begin{cases} G''(\theta) + \lambda G(\theta) = 0 \\ r^2 F''(r) + rF'(r) - \lambda F(r) = 0 \end{cases}$$

$\lambda = m^2$

$$\implies \begin{cases} G(\theta) = \hat{A}\cos m\theta + \tilde{B}\sin m\theta = Ae^{im\theta} + Be^{-im\theta} \\ F(r) = \begin{cases} C\log r + D & m = 0 \\ Cr^m + Dr^{-m} & m \neq 0 \end{cases} \end{cases}$$

$$\left(F(r)\,r^{-m}\right)' = F'(r)\,r^{-m} - mF(r)\,r^{-(m+1)}$$

$$\left(F(r)\,r^{-m}\right)'' = F''(r)\,r^{-m} - 2mF'(r)\,r^{-(m+1)} + m(m+1)F(r)\,r^{-(m+2)}$$

$$r\left(F(r)\,r^{-m}\right)'' + (2m+1)\left(F(r)\,r^{-m}\right)' = 0$$

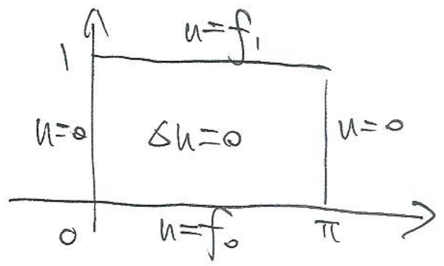
$$r\left(F(r)\,r^{-m}\right)' + 2m\left(F(r)\,r^{-m}\right) = c$$

$\Rightarrow$ we reject r^{-m} $(m>0)$ since as $r \to 0$, $r^{-m} \to +\infty$

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}$$

$$u(1,\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \stackrel{?}{=} f(\theta)$$

Problem 1.



$$\begin{cases} \Delta u = 0, \quad \text{in } R, \\ u(0,y) = u(\pi,y) = 0, \quad 0<y<1, \\ u(x,0) = f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx, \quad 0<x<1, \\ u(x,1) = f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx, \quad 0<x<1, \end{cases} \quad \Rightarrow u(x,y) = ?$$

Sol. Suppose that $u(x,y) = g(x)h(y)$. then $\Delta u = 0$ implies that

$$-\frac{g''(x)}{g(x)} = \frac{h''(y)}{h(y)} = k^2 .$$

Therefore, $g(x) = \sin kx$ or $\cos kx$.

$h(y) = e^{kx}$ or e^{-kx}

Since $u(0,y) = u(\pi,y) = 0$, then $u(x,y)$ is in the form

$$u(x,y) = \sum_{k=1}^{\infty} \left(\tilde{A}_k e^{ky} + \tilde{B}_k e^{-ky} \right) \sin kx .$$

where $\begin{cases} \tilde{A}_k + \tilde{B}_k = A_k \\ \tilde{A}_k e^{k} + \tilde{B}_k e^{-k} = B_k \end{cases}$

$\square$