Solution to Assignment 5

Ex 11. (p. 90) *Check the notes of Tutorial 5.*

Ex 15. (p. 92) (a)&(b) *Check the notes of Tutorial 5.*

(c) Since $f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}$, we have that

$$\begin{aligned}
|f(x+h) - f(x) &= \left| \sum_{k=0}^{\infty} 2^{-k\alpha} (e^{i2^k(x+h)} - e^{i2^k x}) \right| \\
&\leq \sum_{2^k \leq 1/|h|} 2^{-k\alpha} |e^{i2^k(x+h)} - e^{i2^k x}| + \sum_{2^k > 1/|h|} 2^{-k\alpha} |e^{i2^k(x+h)} - e^{i2^k x}| \\
&:= I_1 + I_2.
\end{aligned}$$

For I_1, if $|h| > 1$, then $I_1 = 0$; if $|h| < 1$, then it follows from the Mean Value Theorem that

$$\begin{aligned}
I_1 &= \sum_{2^k \leq 1/|h|} 2^{-k\alpha} |e^{i2^k h} - e^0| \leq \sum_{2^k \leq 1/|h|} 2^{-k\alpha} 2^k |h| \\
&\leq |h| \sum_{k=0}^{k_0} 2^{k(1-\alpha)}, \qquad \text{where } k_0 = \left[\log_2 \frac{1}{|h|} \right] \\
&\leq |h| \frac{2^{(k_0+1)(1-\alpha)} - 1}{2^{1-\alpha} - 1} \leq |h| \frac{\left(\frac{1}{|h|}\right)^{1-\alpha} 2^{1-\alpha}}{2^{1-\alpha} - 1} \leq |h|^{\alpha}.
\end{aligned}$$

For I_2,

$$I_2 \leq \sum_{k=k_0+1}^{\infty} 2^{-k\alpha} \cdot 2 \leq \frac{2^{-(k_0+1)\alpha+1}}{1 - 2^{-\alpha}} \leq \frac{2|h|^{\alpha}}{1 - 2^{-\alpha}}.$$

Therefore, $|f(x+h) - f(x) \leq C|h|^{\alpha}$ for some $C > 0$ independent of x. However, it is easy to check that $\hat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$, which implies that the result in (b) cannot be improved. $\square$

Ex 16. (p. 92) (a)Since f is a $2\pi-$periodic function, by changing of coordinates we have

$$\begin{aligned}
\hat{g_h}(n) &= \frac{1}{2\pi} \int_0^{2\pi} (f(x+h) - f(x-h)) e^{-inx} \, dx \\
&= \int_h^{2\pi+h} f(x) e^{-in(x-h)} \, dx - \int_{-h}^{2\pi-h} f(x) e^{-in(x+h)} \, dx \\
&= \int_0^{2\pi} f(x) e^{-inx} (e^{inh} - e^{-inh}) \, dx = (2i \sin nh) \hat{f}(n).
\end{aligned}$$

Therefore, it follows from the Parseval's Identity that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |\hat{g_h}(n)|^2 = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2|\hat{f}(n)|^2.$$

Since f is Lipschitz continuous, $|g_h| \leq 2Kh$. Then the above identity implies that

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2|\hat{f}(n)|^2 \leq \frac{1}{4}\int_0^{2\pi} (2Kh)^2\, dx = K^2h^2$$

(b) When $h = \pi/2^{p+1}$ and $2^{p-1} \leq |n| \leq 2^p$, $\frac{\pi}{4} \leq |nh| \leq \frac{\pi}{2}$ and then $|\sin nh| \geq \sqrt{2}/2$. Therefore,

$$\sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)|^2 \leq 2\sum_{2^{p-1}\leq|n|\leq 2^p} |\sin nh|^2|\hat{f}(n)|^2 \leq 2K^2h^2 = \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) It follows the Cauchy-Schwarz inequality that

$$\begin{aligned}\left(\sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)|\right)^2 &\leq \sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)|^2 \cdot \sum_{2^{p-1}\leq|n|\leq 2^p} 1^2 \\ &\leq \frac{K^2\pi^2}{2^{2p+1}}2^p = \frac{K^2\pi^2}{2^{p+1}}.\end{aligned}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{p=1}^{\infty}\sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)| \leq |\hat{f}(0)| + \sum_{p=1}^{\infty}\frac{K\pi}{2^{(p+1)/2}} < \infty,$$

which implies that the Fourier series of f converges absolutely and uniformly.

(d) If f is Hölder continuous of order $\alpha > 1/2$, then $|g_h(x)| \leq C(2h)^\alpha$ for some constant $C > 0$. And similarly we have

$$\sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)| \leq \sqrt{2C^2(2h)^{2\alpha}\cdot 2^p} = C\pi^\alpha 2^{p(\frac{1}{2}-\alpha)-\frac{1}{2}},$$

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{p=1}^{\infty}\sum_{2^{p-1}\leq|n|\leq 2^p} |\hat{f}(n)| \leq |\hat{f}(0)| + \sum_{p=1}^{\infty} C\pi^\alpha 2^{p(\frac{1}{2}-\alpha)-\frac{1}{2}} < \infty.$$

Note that here $2^{\frac{1}{2}-\alpha} < 1$ for $\alpha > 1/2$. Therefore, the Fourier series of f converges absolutely. □