Solution to Assignment 3

9(a) (p.61) It is easy to see $\widehat{f}(0) = \frac{b-a}{2\pi}$. If $n \neq 0$,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{a}^{b} e^{-inx}dx = \frac{e^{-ina} - e^{-inb}}{2\pi in}.$$

Hence,

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.$$

9(b). The Fourier series does not converge absolutely means that we need to show

$$\sum_{n\neq 0} \left| \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx} \right| = \infty.$$

Denote $\theta_0 = \frac{b-a}{2}$,

$$\begin{aligned} \sum_{n\neq 0} \left| \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx} \right| &= \sum_{n\neq 0} \left| e^{-in(b+a)/2} \, \frac{e^{in(b-a)/2} - e^{-in(b-a)/2}}{2\pi in} \right| \\ &= \sum_{n\neq 0} \left| \frac{\sin n\theta_0}{\pi n} \right|. \end{aligned} \tag{0.1}$$

Note that from the assumption, $\theta_0 < \pi$. Hence, we can find some $c > 0$ so that

$$\frac{\pi - 2\sin^{-1} c}{\theta_0} > 1.$$

This means for all integers $k \geq 1$, the length of the intervals $(\frac{\pi k + \sin^{-1} c}{\theta_0}, \frac{\pi(k+1) - \sin^{-1} c}{\theta_0})$ is equal to $\frac{\pi - 2\sin^{-1} c}{\theta_0} > 1$. This implies there exists some integer n_k such that

$$n_k\theta_0 \in (\pi k + \sin^{-1} c, \pi(k+1) - \sin^{-1} c).$$

This means that $n_k \leq \frac{\pi(k+1) - \sin^{-1} c}{\theta_0} \leq \frac{\pi(k+1)}{\theta_0}$ and $|\sin n_k\theta_0| \geq c$. Hence,

$$(0.1) \geq \sum_{n>0} \left| \frac{\sin n\theta_0}{\pi n} \right| \geq \sum_{k=1}^{\infty} \left| \frac{\sin n_k\theta_0}{\pi n_k} \right| \geq \sum_{k=1}^{\infty} \frac{c}{\pi \frac{\pi(k+1)}{\theta_0}} = \frac{\theta_0}{c\pi^2} \sum_{k=1}^{\infty} \frac{1}{k+1}.$$

As the series $\sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$, the proof is completed.
Remark. *Drawing a graph of $y = |\sin x|$ helps visualizing the argument.*

9(c). Note that

$$\begin{aligned} & \frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}+\frac{e^{-i(-n)a}-e^{-i(-n)b}}{2\pi i(-n)}e^{i(-n)x} \\ = & \frac{1}{2\pi in}\left[(e^{in(x-a)}-e^{-in(x-a)})-(e^{in(x-b)}-e^{-in(x-b)})\right] \\ = & \frac{1}{\pi n}(\sin n(x-a)-\sin n(x-b)) \end{aligned}$$

The Fourier series of f becomes

$$\frac{b-a}{2\pi}+\sum_{n\geq 1}\frac{1}{\pi n}(\sin n(x-a)-\sin n(x-b)).$$

By the Dirichlet's Test (Ex 7b (p.60)), $\sum_{n\geq 1}\frac{\sin n(x-a)}{n}$ and $\sum_{n\geq 1}\frac{\sin n(x-b)}{n}$ converge for all x. Hence the Fourier series converges at every point x.

If $a=-\pi$ and $b=\pi$, then $\widehat{f}(n)=0$ for $n\neq 0$, then the Fourier series of f is $\frac{b-a}{2\pi}\equiv 1$ is equal to f itself.

15 (p.63). Let $\omega=e^{ix}$, then $D_k(x)=\sum_{n=-k}^{n=k}(e^{ix})^n=\frac{\omega^{-k}-\omega^{k+1}}{1-\omega}$.

$$\begin{aligned} NF_N(x) &= \sum_{k=0}^{N-1} D_k(x) \\ &= \sum_{k=0}^{N-1}\frac{\omega^{-k}-\omega^{k+1}}{1-\omega} \\ &= \frac{\omega}{(1-\omega)^2}(\omega^N+\omega^{-N}-2) \\ &= \frac{1}{(\omega^{1/2}-\omega^{-1/2})^2}(\omega^{N/2}-\omega^{-N/2})^2 \\ &= \frac{\sin^2(Nx/2)}{\sin^2(x/2)}. \end{aligned}$$

Hence, $F_N(x)=\frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$.

17(a) (p.63) Note that $P_r(\theta)$ is an even function of θ,

$$\frac{1}{2\pi}\int_{-\pi}^{0}P_r(\theta)d\theta=\frac{1}{2\pi}\int_{0}^{\pi}P_r(\theta)d\theta=\frac{1}{2}.$$

We then decompose $A_rf(\theta)$ as

$$A_rf(\theta)=\frac{1}{2\pi}\int_{-\pi}^{0}f(\theta-\varphi)P_r(\varphi)d\varphi+\frac{1}{2\pi}\int_{0}^{\pi}f(\theta-\varphi)P_r(\varphi)d\varphi.$$

We consider the second term. Using standard good kernel argument with $P_r(\theta)$ is a good kernel (Lemma 5.5 p.55), for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(\theta-\varphi) - f(\theta^-)| < \epsilon$. Given this $\delta > 0$, there exists r_0 such that for all $1 > r \geq r_0$,

$0 < \varphi < \delta$

$$\int_\delta^\pi P_r(\varphi)d\varphi < \epsilon.$$

With $P_r(\theta)$ is non-negative and f is bounded on $\mathbb{T}$ (since f is Riemann integrable), we see that

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_0^\pi f(\theta-\varphi)P_r(\varphi)d\varphi - \frac{f(\theta^-)}{2}\right| &= \left|\frac{1}{2\pi}\int_0^\pi (f(\theta-\varphi)-f(\theta^-))P_r(\varphi)d\varphi\right| \\
&\leq \frac{1}{2\pi}\int_0^\delta |f(\theta-\varphi)-f(\theta^-)|P_r(\varphi)d\varphi \\
&\quad + \frac{1}{2\pi}\int_\delta^\pi |f(\theta-\varphi)-f(\theta^-)|P_r(\varphi)d\varphi \\
&< \frac{1}{2\pi}\int_0^\delta \epsilon P_r(\varphi)d\varphi + \frac{2B}{2\pi}\int_\delta^\pi P_r(\varphi)d\varphi \\
&< \epsilon + \frac{B}{\pi}\epsilon,
\end{aligned}$$

where B is the bound of f. This shows

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^\pi f(\theta-\varphi)P_r(\varphi)d\varphi = \frac{f(\theta^-)}{2}.$$

Similarly,

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^0 f(\theta-\varphi)P_r(\varphi)d\varphi = \frac{f(\theta^+)}{2}.$$

Hence, $\lim_{r\to 1} A_r f(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}$.

17(b). Since the Fejér kernel is also a good kernel and it is an even function of θ, we can prove the result by applying the same procedure as in (a). We therefore omit the detail.