THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Homework 1

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1. (2 points) Let $S = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$, find $\sup S$ and $\inf S$. Justify it.

Solution: For even natural number n, we have that

$$1 \leq 1 + \frac{(-1)^n}{n} = 1 + \frac{1}{n} \leq \frac{3}{2}.$$

For odd natural number n, we have that

$$0 \leq 1 + \frac{(-1)^n}{n} = 1 - \frac{1}{n} \leq 1.$$

In sum, we can conclude that for all $n \in \mathbb{N}$,

$$0 \leq 1 + \frac{(-1)^n}{n} \leq \frac{3}{2}.$$

Then $\frac{3}{2}$ is an upper bound of S. Let u be an upper bound of S. Then we have that $u \geq 1 + \frac{(-1)^n}{n} = \frac{3}{2}$ when $n = 2$. It follows that $\sup S = \frac{3}{2}$.

Similarly, from previous computation we know that 0 is a lower bound of S. Let v be a lower bound of S. Then we have that $v \leq 1 + \frac{(-1)^n}{n} = 0$ when $n = 1$. It follows that $\inf S = 0$.

2. (2 points) Let S be a non-empty subset of $\mathbb{R}$. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the following holds: For any $t \in \mathbb{R}$, $t > u$ implies $t \notin S$.

Solution:

(a) Suppose that $u \in \mathbb{R}$ is an upper bound of S. Let $t \in \mathbb{R}$. We assume that $t \in S$, then by defintion of an upper bound, we have that $t \leq u$. Hence $t > u$ implies $t \notin S$ for any $t \in \mathbb{R}$.

(b) Suppose that for any $t \in \mathbb{R}$, $t > u$ implies $t \notin S$. Then $t \in S$ implies $t \leq u$ for any $t \in \mathbb{R}$. Therefor we conclude that u is an upper bound of S.

3. (3 points) Show that if A, B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded subset and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Solution: Since A, B are bounded, we can find $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that for any $a \in A$ and $b \in B$,

$$a_1 \leq a \leq a_2, \quad b_1 \leq b \leq b_2.$$

It follows that for any $x \in A \cup B$,

$$\min\{a_1, b_1\} \leq x \leq \max\{a_2, b_2\}.$$

Therefore $A \cup B$ is bounded.

Let $a_2 = \sup A$ and $b_2 = \sup B$ in previous computation. Then we can conclude that $A \cup B$ is bounded above by $\max\{\sup A, \sup B\}$. Suppose $A \cup B$ is bounded above by some $u \in \mathbb{R}$. Then for any $a \in A \subset A \cup B$ and $b \in B \subset A \cup B$,

$$a \leq u \text{ and } b \leq u.$$

Hence A, B are both bounded above by u. It follows that $\sup A \leq u$ and $\sup B \leq u$. Therefore, $\max\{\sup A, \sup B\} \leq u$, which implies that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

4. (3 points) Let S be a bounded subset of $\mathbb{R}$ and S_0 be a non-empty subset of S. Show that
$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$

Solution: Since S_0 is non-empty, by picking $a \in S_0$, we have that
$$\inf S_0 \leq a \leq \sup S_0.$$
For any $s \in S_0$, since $S_0 \subset S$, we have that $s \in S$ and
$$\inf S \leq s \leq \sup S.$$
Hence S_0 is bounded above by $\sup S$ and bounded below by $\inf S$. From the definition of $\sup S_0$ and $\inf S_0$, we can conclude that
$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$