THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Tutorial 3
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Problems:

1. Let G be a group and H a subgroup. Show that $C(H) = \{g \in G : gh = hg, \forall h \in H\}$ is a subgroup of G. This subgroup $C(H)$ is called the centralizer of H in G.

Solution.
- Associativity follows from that in G.
- Clearly $e \in C(H)$.
- Let $x, y \in C(H)$, then for any $h \in H$,

$$(xy)h = x(yh) \xlongequal{\because\, y \in C(H)} x(hy) = (xh)y \xlongequal{\because\, x \in C(H)} (hx)y = h(xy).$$

So $xy \in C(H)$. So $C(H)$ is closed.
- When $x \in C(H)$, we have, for any $h \in H$,

$$xh = hx \quad \Rightarrow \quad hx^{-1} = x^{-1}h.$$

So $x^{-1} \in C(H)$.

For what H does $H \subset C(H)$ happen? Answer: H is abelain. Observe the converse is also true!

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2. Find all the subgroups of Klein 4-group.

Solution. Note that Klein 4-group $= \{(0,0),(0,1),(1,0),(1,1)\}$. Clearly $\{(0,0)\}$ and $\{(0,0),(0,1),(1,0),(1,1)\}$ are trivial subgroups. One may check directly that

$$\langle(0,1)\rangle = \{(0,0),(0,1)\},$$

$$\langle(1,0)\rangle = \{(0,0),(1,0)\}$$

and

$$\langle(1,1)\rangle = \{(0,0),(1,1)\}$$

are the proper subgroup. These are all the proper subgroups as any two of $(0,1),(1,0),(1,1)$ can form the remaining one: $(0,1)+(1,0) = (1,1)$, $(1,1)+(1,0) = (1,0)$ and $(1,1)+(1,0) = (0,1)$.

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3. Write down the cyclic subgroup $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ of the group $GL(2, \mathbb{R})$.

Solution.
$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

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4. Show that the alternating group A_n is nonabelian for $n \geq 4$.

Solution. Note that $\begin{pmatrix}1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}$ is even. A_4 is not abelian since
$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \neq \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} = \begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}.$$

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5. Show that a group with no proper nontrivial subgroups is cyclic.

Solution. Let G be a group with no proper nontrivial subgroups. If $G = \{e\}$, then G is of course cyclic. If $G \neq \{e\}$, then there is $a \in G$, such that $a \neq e$. We know that $\langle a \rangle$ is a subgroup of G and $\langle a \rangle \neq \{e\}$. Because G has no proper nontrivial subgroups, we must have $\langle a \rangle = G$, so G is indeed cyclic.

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6. Show that a finite group is never a union of two of its proper subgroups, that is, for any G, $G \neq H_1 \cup H_2$ for some proper subgroups $H_1 < G$ and $H_2 < G$ with $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$. Does the statement remain true if *two* is replaced by *three*?

Solution. Seeking a contradiction, we let H_1 and H_2 be two proper subgroups of G such that $H_1 \cup H_2 = G$. First of all, we have $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$ for otherwise G is equal to its proper subgroup H_1 or H_2. So there are two elements $a \in H_1$ but not in H_2 and $b \in H_2$ but not in H_1. As $H_1 \cup H_2 = G$, $ab \in H_1 \cup H_2$. It follows that $ab \in H_1$ or $ab \in H_2$. For the former one, it is absurd as (since a^{-1} and ab are both in H_1)
$$b = a^{-1}(ab) \in H_1$$
contradicting the choice of b; for the latter, it is also impossible. In each case we have a contradiction.

No. Note that the Klein 4-group is the union of $\langle(0,1)\rangle$, $\langle(1,0)\rangle$, and $\langle(1,1)\rangle$.

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Optional Part

1. Give an example of two groups G and H and a subgroup K of $G \times H$ such that K cannot be written as $K = G_1 \times H_1$, where G_1 and H_1 are subgroups of G and H, respectively.

Solution. Take $G = H = \mathbb{Z}_2$. Then we let $K = \{(g, g) : g \in \mathbb{Z}_2\} = \{(0, 0), (1, 1)\}$. Suppose there are such G_1 and H_1, we then have $0, 1 \in G_1$ and $0, 1 \in H_1$. It forces that $G_1 = H_1 = \mathbb{Z}_2$. However $G_1 \times H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \neq K$, as $(0, 1) \in G_1 \times H_1$ but not in K.

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