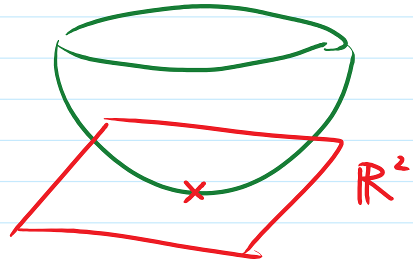
Week 14 Lectures

§ Curvature



- curvature measures deviation of a space from a flat space ($\mathbb{R}^n$).
- curvature measures the noncommutativity of 2nd derivatives, i.e. curvature measures the difference $\dfrac{\partial^2 f}{\partial x \partial y} - \dfrac{\partial^2 f}{\partial y \partial x}$

$\rightsquigarrow$ curvature is about 2nd derivatives

§ Christoffel symbols

Def Given a Riemannian metric $g = (g_{ij})$, the Christoffel symbols Γ_{ij}^k are defined as

$$\Gamma_{ij}^k := \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

where $(g^{kl}) = (g_{ij})^{-1}$.

Rmk These Christoffel symbols describe the coefficients of the Levi-Civita connection associated to g.

In the case of a surface (i.e. when $n=2$), So we only have the following Christoffel symbols

$$\Gamma^1_{11},\ \Gamma^2_{11},\ \underset{\Gamma^1_{21}}{\overset{\Vert}{\Gamma^1_{12}}},\ \underset{\Gamma^2_{21}}{\overset{\Vert}{\Gamma^2_{12}}},\ \Gamma^1_{22},\ \Gamma^2_{22}$$

e.g. In the upper half-plane model of hyperbolic geometry,

$$g = (g_{ij}) = \underline{\begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}}$$

we have

$$\Gamma^1_{11} = 0,\quad \Gamma^2_{11} = \frac{1}{y},\quad \Gamma^1_{12} = -\frac{1}{y},\quad \Gamma^2_{12} = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12},\quad \Gamma^2_{21} = \Gamma^2_{12},\qquad \Gamma^1_{22} = 0,\quad \Gamma^2_{22} = -\frac{1}{y}$$

§ Geodesics (or straight lines)

Using Christoffel symbols, we can write down the equation of the <u>shortest</u> <u>curve</u> (called a geodesic) between two pts. This is done by studying the <u>Euler-Lagrange equation</u> for minimizing $\ell(\gamma) =: \underline{L(\gamma, \gamma')}$

(crit. pt eqn for L)

$$\frac{\partial L}{\partial \gamma} = \frac{d}{dt}\frac{\partial L}{\partial \gamma'}$$

$\longrightarrow$ the equations for a geodesic $\gamma : z(t) = x(t) + iy(t)$ are given by the ODEs:

$$\begin{cases} x''(t) + \Gamma^1_{11} x'(t)^2 + 2\Gamma^1_{12} x'(t) y'(t) + \Gamma^1_{22} y'(t)^2 = 0 \\ y''(t) + \Gamma^2_{11} x'(t)^2 + 2\Gamma^2_{12} x'(t) y'(t) + \Gamma^2_{22} y'(t)^2 = 0 \end{cases}$$

$$\begin{cases} y''(t) + \Gamma^2_{11} x'(t)^2 + 2\Gamma^2_{12} x'(t) y'(t) + \Gamma^2_{22} y'(t)^2 = 0 \end{cases}$$

Rmk: For general dimensions, these eqns look like:

$$\boxed{x_i''(t) + \sum_{j,k} \Gamma^i_{jk} x_j'(t) x_k'(t) = 0}$$

e.g. In the upper half-plane model, these eqns read

$$\begin{cases} x''(t) - \frac{2}{y} x'(t) y'(t) = 0 \\ y''(t) + \frac{1}{y}\left(x'(t)^2 - y'(t)^2\right) = 0 \end{cases}$$

can check: if $x'(t) \neq 0$, then

$$\frac{d}{dt}\left(\frac{y \cdot y' + x \cdot x'}{x'}\right) = 0$$

$$\Longrightarrow \boxed{y \cdot y' + x \cdot x' = c \cdot x'}$$

which is precisely the eqn of a circle with center on $y = 0$

if $x'(t) = 0$, then we have eqn of a vertical line.

Theory of ODEs

$\Rightarrow$ **Thm** When $p, q \in S$ are close enough, $\exists$ a unique geodesic connecting p and q.

§ Curvature

The Gaussian curvature K of a surface is defined in terms of the 1st and 2nd fundamental forms.

A famous theorem of Gauss, known as Theorema Egregium, says that K is actually independent of the 2nd fundamental form and hence is intrinsic.

$\longrightarrow$ K can be expressed in terms of the Christoffel symbols

$$K = \frac{1}{\det(g)} \sum_{m=1}^{2} g_{m2} \left(\frac{\partial \Gamma^m_{11}}{\partial x_2} - \cancel{\frac{\partial \Gamma^m_{12}}{\partial x_1}} + \sum_{\ell=1}^{2} \left(\Gamma^\ell_{11} \Gamma^m_{\ell 2} - \Gamma^\ell_{12} \Gamma^m_{\ell 1} \right) \right)$$

e.g. In the upper half-plane model, we have

$$K = \frac{1}{\frac{1}{y^2} \cdot \frac{1}{y^2}} \cdot \frac{1}{y^2} \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -1 \qquad K : S \to \mathbb{R}$$

$\Longrightarrow$ the hyperbolic geometry $(\mathbb{U}, \bar{H})$ has constant curvature -1

Similarly, you can check that

elliptic geometry has const curvature 1

parabolic geometry has const curvature 0

§ Isometry group

Def An isometry is a transformation $T : S \to S$ which preserves the Riemannian metric g

i.e. $\boxed{T^*(ds^2) = ds^2}$ $\qquad S \xrightarrow{T} S \xrightarrow{a} \mathbb{R}$

(more precisely, if $ds^2 = \underline{a}\, dx^2 + 2b\, dx\, dy + c\, dy^2$

then $T^*(ds^2) = \underline{(a \circ T)}\, dx^2 + \underline{2(b \circ T)}\, dx\, dy + \underline{(c \circ T)}\, dy^2)$

then $T^*(ds^2) = (a\circ T)\,dx^2 + 2(b\circ T)\,dx\,dy + (c\circ T)\,dy^2)$

The group of isometries is denoted as $\mathrm{Isom}(S, g)$

e.g. For the upper half-plane model, we claim

$$\bar{H} = \left\{ T \in M : T \text{ is of the form } T(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$$

preserves $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

i.e. $\bar{H} \subset \mathrm{Isom}(U, ds^2)$ (actually, $\bar{H} = \mathrm{Isom}(U, ds^2)$)

To see this, write

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

Then
$$dx^2 = \left(\frac{dz+d\bar{z}}{2}\right)\cdot\left(\frac{dz+d\bar{z}}{2}\right)$$
$$= \frac{1}{4}\left(dz^2 + 2\,dz\,d\bar{z} + d\bar{z}^2\right)$$
$$dy^2 = -\frac{1}{4}\left(dz^2 - 2\,dz\,d\bar{z} + d\bar{z}^2\right)$$
$$\Rightarrow \quad ds^2 = \frac{dx^2+dy^2}{y^2} = -\frac{4\,dz\,d\bar{z}}{(z-\bar{z})^2}$$

Now given $w = T(z) = \dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$ and $D := ad-bc \neq 0$

we have
$$dw = d\Big(\overbrace{\frac{az+b}{cz+d}}^{z\circ T}\Big)$$
$$= \frac{a(\cancel{cz}+d) - c(\cancel{az}+b)}{(cz+d)^2}\,dz$$

$-\dfrac{4\,dw\,d\bar{w}}{(w-\bar{w})^2}$

$D\,dz$

$$-4\frac{dw\,d\bar{w}}{(w-\bar{w})^2} \qquad (cz+d)$$

$$= \frac{D\,dz}{(cz+d)^2}$$

$$d\bar{w} = \frac{D\,d\bar{z}}{(c\bar{z}+d)^2}$$

Also, note that

$$w-\bar{w} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$$

$$= \frac{(az+b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{(\cancel{acz\bar{z}}+adz+bc\bar{z}+\cancel{bd})-(\cancel{acz\bar{z}}+ad\bar{z}+bcz+\cancel{bd})}{|cz+d|^2}$$

$$= \frac{(ad-bc)z-(ad-bc)\bar{z}}{|cz+d|^2}$$

$$= \frac{D(z-\bar{z})}{|cz+d|^2}$$

$$\Rightarrow \frac{-4\,dw\,d\bar{w}}{(w-\bar{w})^2} = -4\frac{\dfrac{\cancel{D^2}\,dz\,d\bar{z}}{\cancel{|cz+d|^4}}}{\dfrac{\cancel{D^2}(z-\bar{z})^2}{\cancel{|cz+d|^4}}} = \frac{-4\,dz\,d\bar{z}}{(z-\bar{z})^2}. \ \#$$

We already know that

$\bar{H}$ acts transitively on $\mathbb{U}$

(i.e. $\forall\, p, q \in \mathbb{U}$, $\exists\, T \in \bar{H}$ s.t. $T(p) = q$)

More generally, if Isom(S, g) acts transitively on S, then we have

$\Rightarrow$ the geometry looks the same at every pt

$\Rightarrow$ the curvature is the same at every pt

i.e. curvature $\equiv$ const

Conclusion:

Felix Klein's Erlanger Program describes exactly those geometries where the isometry group acts transitively.

§ Riemann's uniformization theorem

This theorem says that every surface equipped any Riemannian metric is "conformally equivalent" to one of the 3 geometries described by Klein namely, elliptic, parabolic or hyperbolic geom.

(curvature $+1$ $\quad$ 0 $\quad$ -1)

- From this thm, one can deduce that every compact surface admits a Riemannian metric with const. curvature

e.g. for a cpt surface of genus 4

$\exists$ a Riemannian metric on this surface with const curvature < 0.

- The Gauss-Bonnet Thm:

genus of S

$$\iint_S K\, ds^2 = \chi(S) = 2 - 2g$$

Euler number of S

(i.e. $V - E + F$)