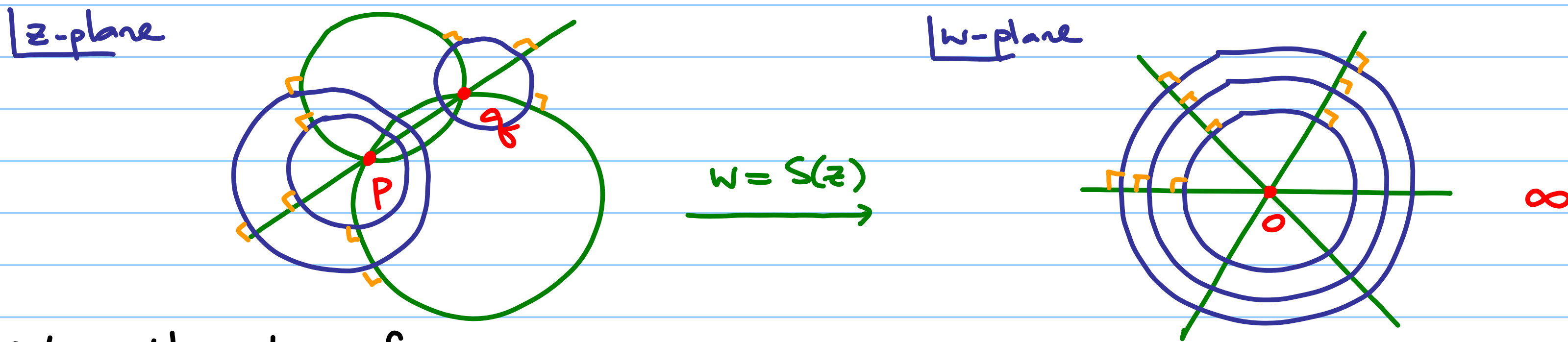
MMAT 5120 Topics in Geometry

Lecture 6

§ Steiner circles (coordinate systems in Möbius geometry)

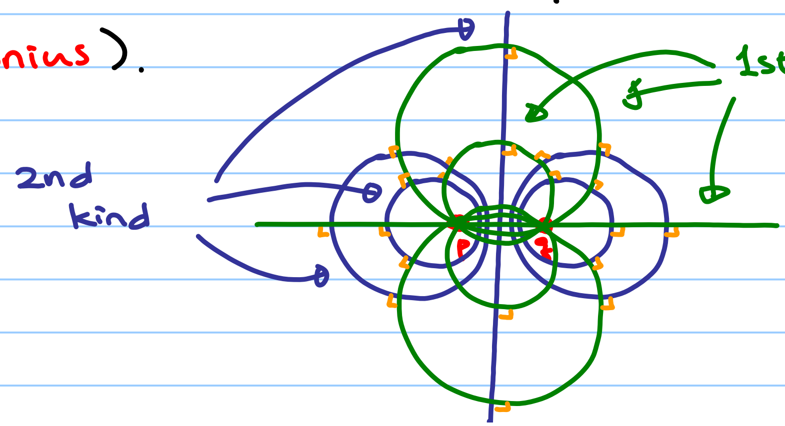
Let $p \neq q \in \mathbb{C}$. The family of all clines passing through p and q is called the Steiner circles of the first kind with respect to p, q.



Consider the transformation

$$w = S(z) = \frac{z-p}{z-q}$$

Then $p \mapsto 0$ and $q \mapsto \infty$ (i.e. $S(p)=0$ and $S(q)=\infty$), and the Steiner circles are mapped to straight lines passing thru 0.
Now in the w-plane, the family of circles centered at $w=0$ is a family of clines orthogonal to the "Steiner circles" of the 1st kind; their pull-backs (or inverse images) by S^{-1} form a family of circles called Steiner circles of the second kind with respect to p, q (also called circles of Apollonius).



By definition, the Steiner circles of 2nd kind w.r.t. p & q are defined by

$$\boxed{\frac{|z-p|}{|z-q|} = k}$$

(as preimage of $\{w \in \mathbb{C} : |w| = k\}$)

<u>Rmk</u> The families of Steiner circles of 1st and 2nd kind can be viewed as a generalization of polar coordinates to Möbius geometry.

Normal form of a Möbius transformation

- Möbius transformations with two fixed pts

Fix $p \neq q$ in $\mathbb{C}$. Let $T \in M$ be a Möbius transformation fixing p & q.
Then T maps clines passing thru p & q to clines passing thru p & q.

$\Longrightarrow$ Steiner circles of 1st kind w.r.t. p & q are invariant under T.

Actually, Steiner circles of 2nd kind are also invariant under T. (Ex.)

To understand the action of T, consider again

$$w = S(z) = \frac{z-p}{z-q}$$

and let $R := S \circ T \circ S^{-1}$ be the lift of T to the w-plane via S^{-1}.

Then $\begin{cases} R(0) = (S\circ T\circ S^{-1})(0) = (S\circ T)(p) = S(p) = 0 \\ R(\infty) = (S\circ T\circ S^{-1})(\infty) = (S\circ T)(q) = S(q) = \infty \end{cases}$

$$\begin{tikzcd} w \in \hat{\mathbb{C}} \arrow[r, "R"] \arrow[d, "S^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "S^{-1}"] \\ z \in \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd}$$

Writing $R(w) = \dfrac{aw+b}{cw+d}$ for $a,b,c,d \in \mathbb{C}$ s.t. $ad-bc \neq 0$,

we see that $R(0)=0 \Rightarrow b=0$ and $R(\infty)=\infty \Rightarrow c=0$

$\therefore$ $R(w) = \lambda w$ where $\lambda = \frac{a}{d} \neq 0$ in $\mathbb{C}$.

Substituting back to $R = S\circ T\circ S^{-1}$, we have

$$\lambda w = (S\circ T\circ S^{-1})(w) \Rightarrow \lambda S(z) = S(T(z))$$

$$\Rightarrow \boxed{\frac{T(z)-p}{T(z)-q} = \lambda \frac{z-p}{z-q}}$$

This is called the **normal form** of T.

Rmk The formula $T = S^{-1} \circ R \circ S$ says that T can be understood as a composition of 3 operations:

(i) sending the two fixed pts p & q to 0 & ∞ resp.

(ii) multiplying by a nonzero scalar $\lambda \in \mathbb{C}^*$

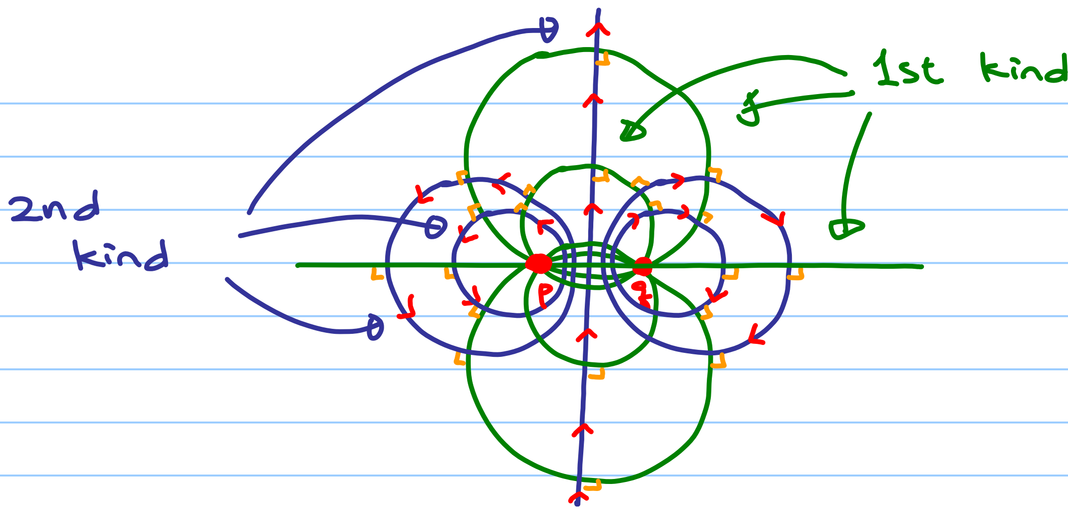
(iii) sending 0 & ∞ back to the fixed pts p & q resp.

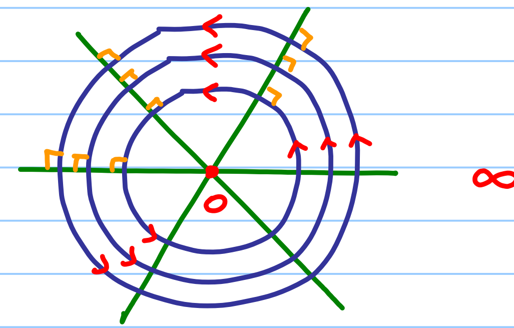
There are 3 cases:

$\boxed{\text{Case 1: } |\lambda| = 1}$ — Eilliptic transformations

Then $\lambda = e^{i\theta}$ and $R(w) = e^{i\theta} w$ is a rotation about the origin.

$\Rightarrow$ T moves pts on the Steiner circles of 2nd kind around, and sends Steiner circles of 1st kind to other Steiner circles of 1st kind.

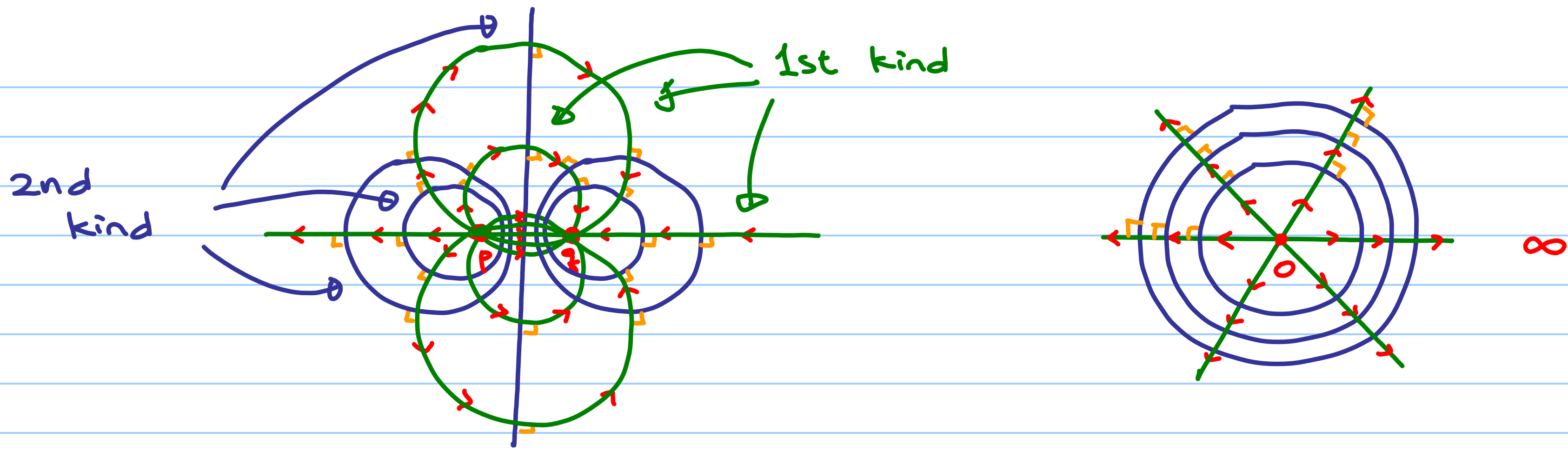




$\boxed{\text{Case 2 : } \lambda \in \mathbb{R}_{>0}}$ – Hyperbolic transformations

Then $R(w) = \lambda w$, $\lambda > 0$ is a homothetic transformation.

$\Rightarrow$ T moves pts on the Steiner circles of 1st kind around, and sends Steiner circles of 2nd kind to other Steiner circles of 2nd kind.



$\boxed{\text{Case 3: } \lambda \notin \mathbb{R} \text{ and } |\lambda| \neq 1}$ – Loxodromic transformations

In this case, $\lambda = ke^{i\theta}$ for some $1 \neq k > 0$ and $\theta \notin 2\pi\mathbb{Z}$.

So $R(w) = \lambda w$ is a composition of a rotation and a homothetic transformation, and T is a composition of an elliptic and a hyperbolic transformation. $\left(T = S^{-1} \circ R_1 \circ R_2 \circ S = (S^{-1} \circ R_1 \circ S) \circ (S^{-1} \circ R_2 \circ S).\right)$

• Möbius transformations with one fixed pt – parabolic transformations

Let $T \in M$ be a Möbius transformation with one fixed pt p.

Consider $$w = S(z) = \frac{1}{z-p}$$

which sends $p \mapsto \infty$. Let $R := S \circ T \circ S^{-1}$ be the lift of T via S^{-1}.

We have $R(\infty) = (S \circ T \circ S^{-1})(\infty) = S(T(p)) = S(p) = \infty$.

Writing $R(w) = \frac{aw+b}{cw+d}$ for $a,b,c,d \in \mathbb{C}$ s.t. $ad-bc \neq 0$.

Then $R(\infty) = \infty \Rightarrow c = 0$, $a \neq 0$ & $d \neq 0 \Rightarrow R(w) = \left(\frac{a}{d}\right)w + \frac{b}{d}$.

Now T has only 1 fixed pt $\Rightarrow R$ has only 1 fixed pt

$\Rightarrow \quad w \neq \left(\frac{a}{d}\right)w + \left(\frac{b}{d}\right) \quad \forall w \in \mathbb{C}$

$\Rightarrow \quad \frac{a}{d} = 1$ and $\frac{b}{d} \neq 0$

We conclude that $R(w) = w + \beta$ for some $0 \neq \beta \in \mathbb{C}$, which is a translation.

Substituting back, we have

$$(S \circ T \circ S^{-1})(w) = R(w) = w + \beta$$

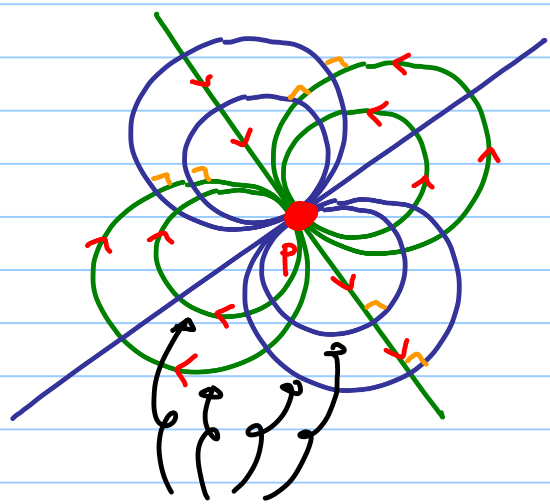
$$\Longrightarrow \quad (S \circ T)(z) = S(z) + \beta$$

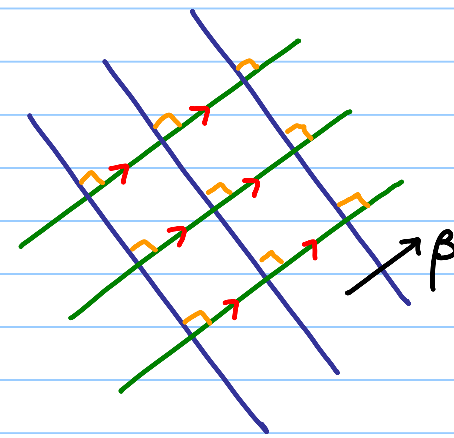
$$\Longrightarrow \quad \boxed{\frac{1}{T(z) - p} = \frac{1}{z - p} + \beta}$$

This gives the normal form of T.

Straight lines parallel and orthogonal to β in the w-plane can be pulled back by S to give a coordinate on the z-plane.

This is called a generalized Cartesian coordinate system.





These are called degenerate Steiner circles.