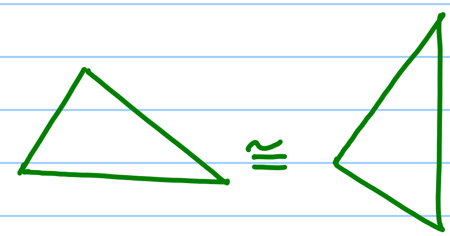
MMAT 5120 Topics in Geometry

Lecture 3

§ The Erlanger Programm

Congruence and transformation

Klein : two figures are congruent when one can be moved by congruence transformations so as to coincide with the other.



So, given two figures A and B, we say they are congruent, denoted as $A \cong B$, iff $A = T(B) := \{T(p) : p \in A\}$ for some congruence transformation T.

The congruence relation $\cong$ should satisfy the following properties:

(a) (reflexivity) $A \cong A$ for any figure A.

(b) (symmetry) If $A \cong B$, then $B \cong A$.

(c) (transitivity) If $A \cong B$ and $B \cong C$, then $A \cong C$.

In other words, $\cong$ should be an equivalence relation.

To achieve this, we require

(a) The identity transformation $\mathrm{Id}(z) \equiv z$ is a congruence transformation.

(b) If T is a congruence transformation, then T is invertible and T^{-1} is also a congruence transformation.

(c) If T_1 and T_2 are congruence transformations, then so is their composition $T_1 \circ T_2$ (defined by $(T_1 \circ T_2)(z) := T_1(T_2(z))$).

This leads to the following definition.

Def Let S be a nonempty set. A transformation group is a collection G of transformations $T: S \to S$ such that

(a) G contains the identity Id_S, i.e. $Id_S \in G$.

(b) $\forall T \in G$, T is invertible and $T^{-1} \in G$.

(c) $\forall T_1, T_2 \in G$, $T_1 \circ T_2 \in G$.

Def A geometry is a pair (S, G) consisting of a nonempty set S and a transformation group G on S. We call S the underlying space of the geometry and G its transformation group.

Def Given a geometry (S, G), a figure is any subset A of the underlying space S. Two figures $A, B \subset S$ are congruent if $\exists T \in G$ s.t. $B = T(A) = \{T(z) : z \in A\}$.

Examples

(1) Plane Euclidean Geometry

underlying space S = complex plane $\mathbb{C}$

transformation group E = the set of transformations of the form

$$T(z) = e^{i\theta} z + b \quad \text{for some } \theta \in \mathbb{R},\ b \in \mathbb{C}.$$

An element $T \in E$, which is a composition of rotation and translation, is called a rigid motion.

The pair $(\mathbb{C}, E)$ models plane Euclidean geometry.

To see that E is a transformation group, we check:

(a) $\mathrm{Id}_{\mathbb{C}} \in E$ is given by taking $\theta = 0$ and $b = 0$.

(b) If $T(z) = e^{i\theta} z + b$, then $T^{-1}(z) = e^{-i\theta}(z - b) = e^{i(-\theta)} + (-e^{-i\theta} b) \in E$.

(c) If $T_1(z) = e^{i\theta_1} z + b_1$ and $T_2(z) = e^{i\theta_2} z + b_2$, then

$$(T_1 \circ T_2)(z) = e^{i\theta_1}(e^{i\theta_2} z + b_2) + b_1 = e^{i(\theta_1 + \theta_2)} z + (b_1 + e^{i\theta_1} b_1) \in E. \quad \#$$

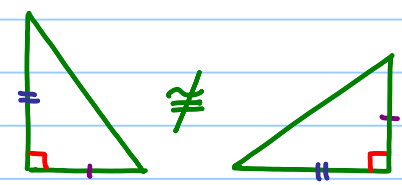
(2) Translational Geometry

underlying space S = complex plane $\mathbb{C}$

transformation group $\mathcal{T} = \{ T : z \mapsto z+b \text{ for some } b \in \mathbb{C} \}$.

Then $(\mathbb{C}, \mathcal{T})$ is a geometry. (Exercise)

In this geometry,



(3) The Trivial Geometry

For any $S \neq \emptyset$, we can take $G = \{Id_S\}$, i.e. G consists of Id_S only.

We call $(S, \{Id_S\})$ the trivial geometry; no two distinct figures are congruent in this geometry!

Invariants

Def Let (S, G) be a geometry. Let D be a set of figures from (S, G), i.e. elements of D are subsets of S.

- The set D is said to be **invariant** in (S,G) if $\forall B \in D$ and $T \in G$, we have $T(B) \in D$.
- A function f defined on D (assumed to be invariant) is called **invariant** if $f(T(B)) = f(B)$ $\forall B \in D$ and $T \in G$.

e.g. In the plane Euclidean geometry $(\mathbb{C}, E)$,

- $D = \{\text{triangles in } \mathbb{C}\}$ is invariant

- Area, perimeter of triangles are invariant functions
- d = sum of distances from vertices to origin is <u>NOT</u> an invariant function.

Klein's Erlanger Programm:

> The proper subject matter of a geometry is its invariant sets and the invariant functions on those sets

Geometric proof

Let (S, G) be a geometry.

If (i) F is an elt in a set D of invariant figures in S such that statement "W" is true, and

(ii) all measurements and quantities mentioned in the statement "W" are invariant,

then $\forall T \in G$, "W" is also true for $T(F)$.

So the strategy for proving that a statement "W" is true for F is to find $T \in G$ so that "W" is easy to see for $T(F)$.

Abstract geometries and their models

Def Two geometries (S_1, G_1) and (S_2, G_2) are models of the same abstract geometry if $\exists$ an invertible covering transformation $\mu : S_1 \longrightarrow S_2$ such that

$$\begin{cases} T_1 \in G_1 \Rightarrow \mu \circ T_1 \circ \mu^{-1} \in G_2 \\ T_2 \in G_2 \Rightarrow \mu^{-1} \circ T_2 \circ \mu \in G_1 \end{cases}$$

$$\begin{tikzcd} S_1 \arrow[r, "T_1"] \arrow[d, "\mu"'] & S_1 \arrow[d, "\mu"] \\ S_2 \arrow[r, "\mu \circ T_1 \circ \mu^{-1}"'] & S_2 \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, "\mu^{-1} \circ T_2 \circ \mu"] \arrow[d, "\mu"'] & S_1 \arrow[d, "\mu"] \\ S_2 \arrow[r, "T_2"'] & S_2 \end{tikzcd}$$

In this case, we say (S_1, G_1) and (S_2, G_2) are isomorphic and μ is an isomorphism.

e.g. Consider the geometries

$S_1 = \{z \in \mathbb{C} : |z| < 1\}$, $G_1 = \{$rotations around the origin$\}$,

$S_2 = \{z \in \mathbb{C} : |z-5| < 3\}$, $G_2 = \{$rotations around $z = 5\}$.

(Exercise: check they are really geometries)

We claim that the map $\mu : S_1 \to S_2$, $z \mapsto 3z + 5$ defines an isomorphism between (S_1, G_1) and (S_2, G_2).

First of all, μ^{-1} exists and is given by $w \mapsto \frac{w-5}{3}$.

Now, let $T_1 \in G_1$. Then $T_1(z) = e^{i\theta_1} z$ for some $\theta_1 \in \mathbb{R}$.

Then, $\forall\, w \in S_2$, we have

$$(\mu \circ T_1 \circ \mu^{-1})(w) = (\mu \circ T_1)(\mu^{-1}(w)) =$$

$$= (\mu \circ T_1)\left(\frac{w-5}{3}\right)$$

$$= \mu\left(e^{i\theta_1}\left(\frac{w-5}{3}\right)\right)$$

$$= 3e^{i\theta_1}\left(\frac{w-5}{3}\right) + 5$$

$$= e^{i\theta_1}(w-5) + 5 \in G_2$$

Similarly, one can check that $\mu^{-1} \circ T_2 \circ \mu \in G_1 \ \forall\, T_2 \in G_2$. #