MMAT 5120 Topics in Geometry

Lecture 2

§ Geometric transformations (cont'd)

The inversion

$$T : \mathbb{C}^* := \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}^*, \quad z \mapsto \frac{1}{z}$$

has several interesting properties:

① Points inside the unit circle are transformed to points outside the unit circle, and vice versa

Pf: If $|z| < 1$, then $|T(z)| = 1/|z| > 1$.
Similarly, $|z| > 1 \Rightarrow |T(z)| < 1$. #

② Points in the upper half plane are transformed to points in the lower half plane, and vice versa

Pf : Since $\arg T(z) = -\arg z$, so $0 < \arg z < \pi \Rightarrow -\pi < \arg T(z) < 0$ and $-\pi < \arg z < 0 \Rightarrow 0 < \arg T(z) < \pi$. #

③ The inversion transforms a straight line passing through 0 to a straight line passing through 0, and a straight line not passing through 0 to a circle.

Pf : A straight line is given by

$$ax + by + c = 0$$

where $a, b, c \in \mathbb{R}$ are consts (and a, b not both zero).

The inversion T is written in coordinates as

$$s + it = w = T(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

i.e. $s = \dfrac{x}{x^2+y^2}$, $t = \dfrac{-y}{x^2+y^2}$

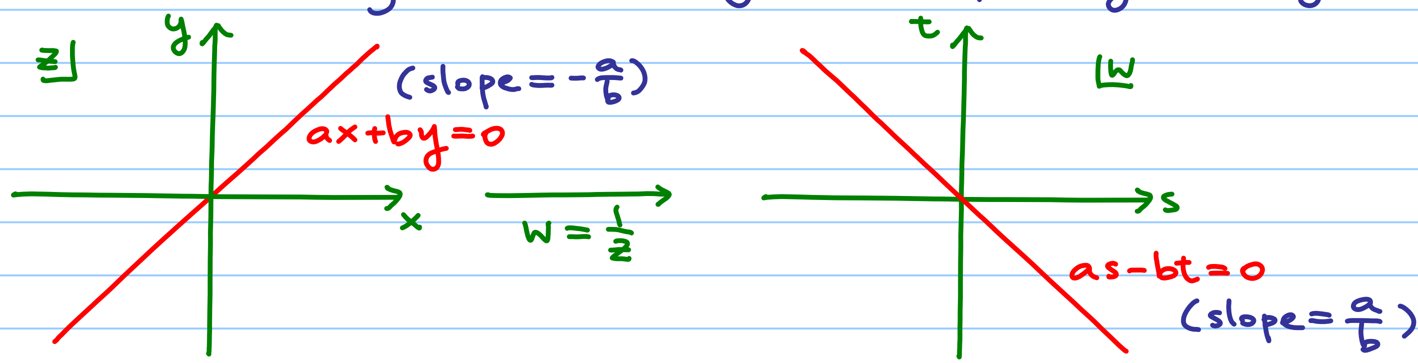
So we have

$$\begin{cases} \text{(i)} & s^2+t^2 = \dfrac{1}{x^2+y^2} \quad \left(\Longleftrightarrow \ |w|^2 = \dfrac{1}{|z|^2} \right) \\ \text{(ii)} & as-bt = -c(s^2+t^2) \end{cases}$$

Case 1 : If the straight line passes through 0, then $c=0$

$\Longrightarrow \quad as-bt=0$

So the image is a straight line passing through 0.

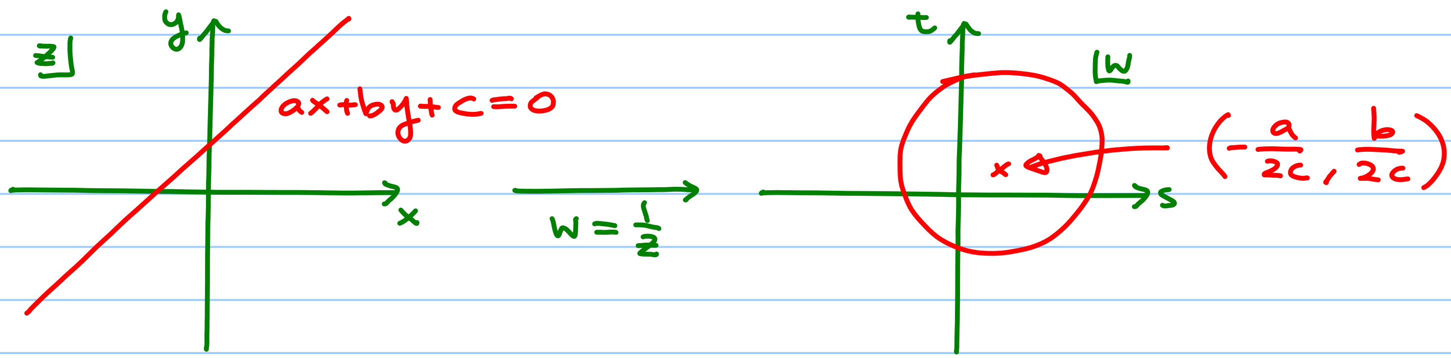


Case 2: If the straight line doesn't pass thru 0, then $c \neq 0$.

$$\Longrightarrow \quad s^2 + t^2 + \left(\frac{a}{c}\right)s - \left(\frac{b}{c}\right)t = 0$$

$$\Longrightarrow \quad \left(s + \frac{a}{2c}\right)^2 + \left(t - \frac{b}{2c}\right)^2 = \frac{a^2+b^2}{4c^2}$$

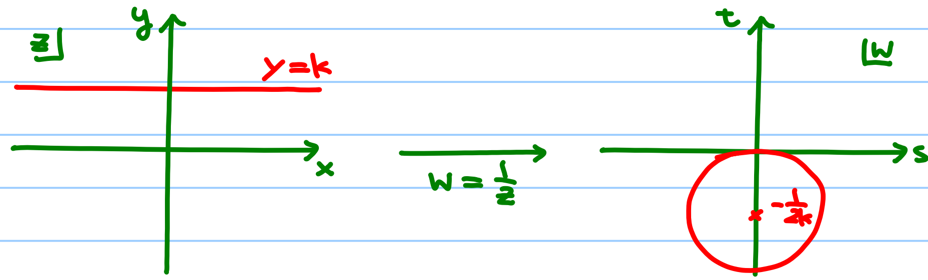
So the image is a circle centered at $\left(-\frac{a}{2c}, \frac{b}{2c}\right)$ with radius $\frac{\sqrt{a^2+b^2}}{2|c|}$. #



Special cases to help visualizing the inversion:

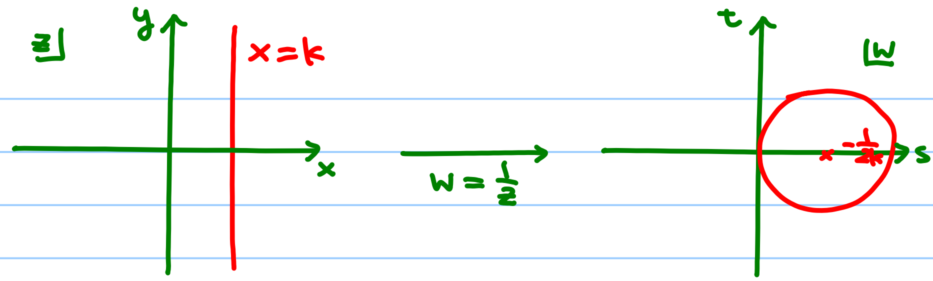
- Horizontal lines $y = k$ (i.e. $a=0$, $b=1$, $c=-k \neq 0$)

$$\Longrightarrow \text{circle}: \quad s^2 + \left(t + \frac{1}{2k}\right)^2 = \left(\frac{1}{2k}\right)^2$$

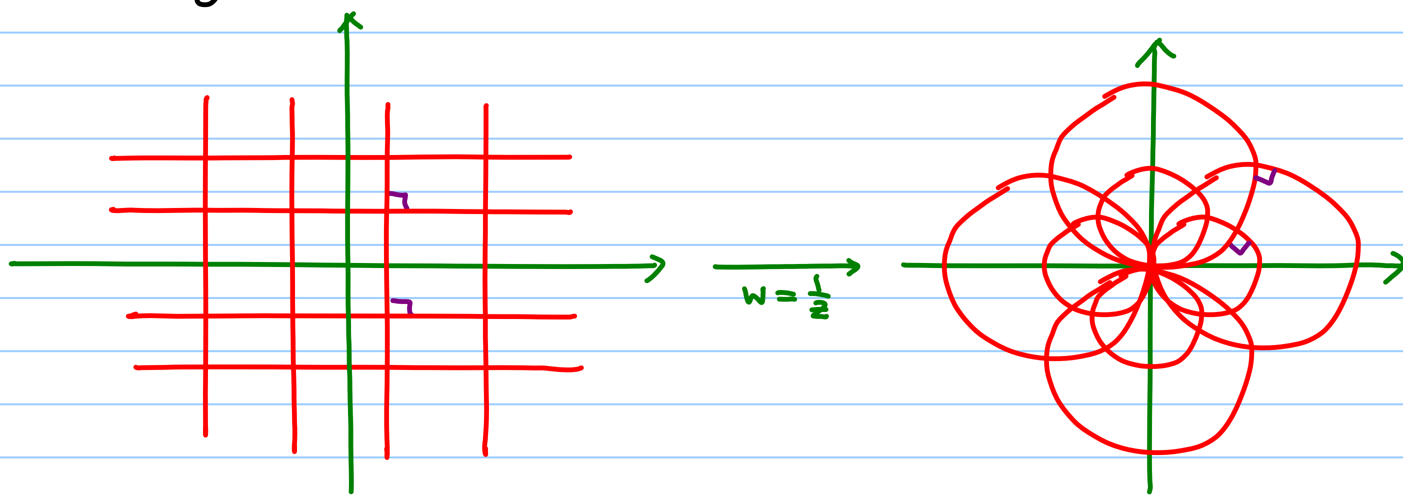


- Horizontal lines $x = k$ (i.e. $a=1$, $b=0$, $c=-k \neq 0$)

$$\Longrightarrow \text{circle}: \quad \left(s - \frac{1}{2k}\right)^2 + t^2 = \left(\frac{1}{2k}\right)^2$$

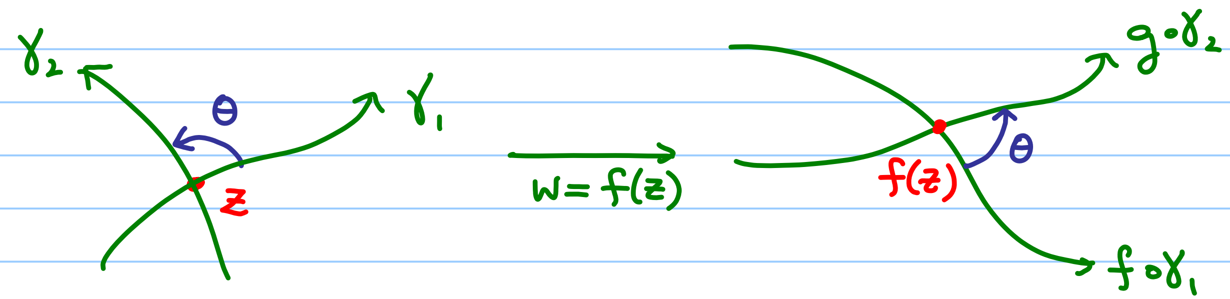


So altogether we have



Conformality

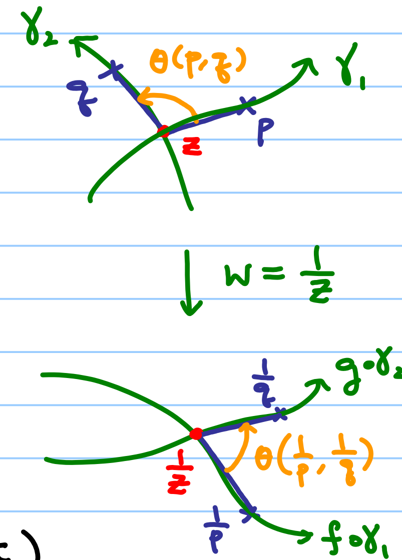
A transformation f is called conformal if it preserves angles, i.e. if γ_1, γ_2 are curves passing thru a pt z, then the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z)$ is equal to the angle between γ_1 and γ_2 at z.



e.g. Translations, rotations and homothetic transformations are all conformal.

Claim : The inversion is conformal at every $z \in \mathbb{C}^*$

1st pf : Referring to the figures on the right,



$$\theta(p,q) = \arg \frac{q-z}{p-z}$$

and

$$\theta\left(\frac{1}{p}, \frac{1}{q}\right) = \arg \frac{\frac{1}{q} - \frac{1}{z}}{\frac{1}{p} - \frac{1}{z}}$$

$$= \arg \frac{p}{q} \cdot \frac{q-z}{p-z}$$

$$= \arg \frac{p}{q} + \theta(p,q) \quad (\text{mod } 2\pi)$$

If $p, q \to z$, we have $\frac{p}{q} \to \frac{z}{z} = 1$, so $\arg \frac{p}{q} \to 0$.

Hence $\lim\limits_{p,q\to z} \theta\left(\frac{1}{p}, \frac{1}{q}\right) = \lim\limits_{p,q\to z} \theta(p,q)$. #

2nd pf: $\frac{d}{dz}T(z) = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \neq 0 \Rightarrow T$ is conformal $\forall z \in \mathbb{C}^*$. #

<u>Rmk</u> In general, we have the following thm from complex analysis (MMAT5220)

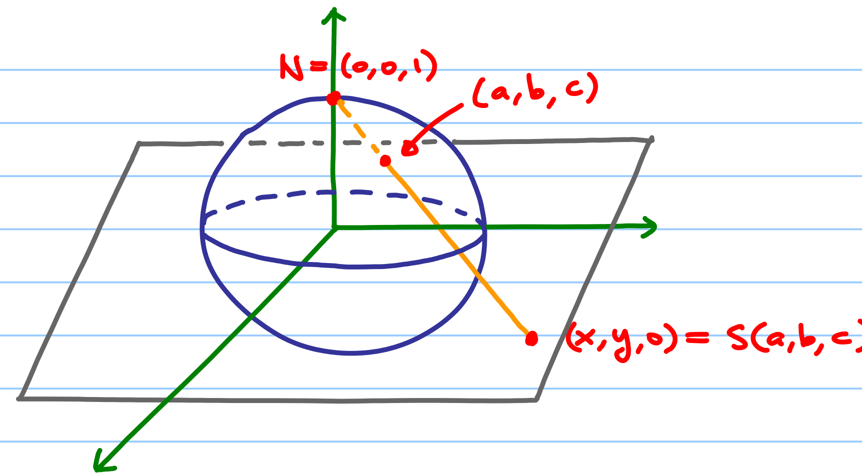
<u>Thm</u> A function f is conformal at z_0 iff f is complex differentiable at z_0 and $f'(z_0) \neq 0$.

Stereographic projection

This is a map

$$S : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$$

where $S^2 = \{(a,b,c) \in \mathbb{R}^3 : a^2+b^2+c^2 = 1\}$



Claim : $S(a,b,c) = x + iy = \dfrac{a+ib}{1-c}$

Pf : The straight line passing thru the north pole $N=(0,0,1)$ and (a,b,c) can be written as

$$(0,0,1) + t\left[(a,b,c) - (0,0,1)\right], \quad t \in \mathbb{R}$$

It intersects the xy-plane when $1 + t(c-1) = 0 \iff t = \dfrac{1}{1-c}$

$$\Rightarrow \quad x = t \cdot a = \frac{a}{1-c}, \quad y = t \cdot b = \frac{b}{1-c}. \quad \#$$

<u>Rmk</u> The stereographic projection is conformal. (See the book for a rough argument.)

<u>Point at ∞</u>: In view of the stereographic projection, it is natural to consider adding a point ∞ to $\mathbb{C}$ so that we have a bijective correspondence

$$S : S^2 \longrightarrow \mathbb{C} \cup \{\infty\}$$

so that $N \longleftrightarrow \infty$ and $(a,b,c) \to (0,0,1) \Leftrightarrow S(a,b,c) \to \infty$.

Note that $|S(a,b,c)| = \sqrt{\dfrac{1+c}{1-c}}$ since $a^2+b^2+c^2=1$.

So $(a,b,c) \to (0,0,1) \Leftrightarrow |S(a,b,c)| \to +\infty$.

<u>Rmk</u> $\mathbb{C} \cup \{\infty\}$ is denoted as $\hat{\mathbb{C}}$ or $\mathbb{CP}^1$ and called the extended complex plane.

e.g. The inversion $T : z \mapsto \frac{1}{z}$ can actually be extended to a transformation on $\hat{\mathbb{C}}$ by

$$z \mapsto \frac{1}{z} \quad \forall z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$
$$0 \mapsto \infty,$$
$$\infty \mapsto 0.$$

<u>Def</u> Let $S : D \to R$ be a surjective continuous map.

- We say that S is a covering transformation from D to R or that D covers R.
- Let $f : R \to R$ be a transformation. A transformation $g : D \to D$ is called a lift of f if $S(g(z)) = f(S(z))\ \forall z \in D$, i.e. the following diagram commutes

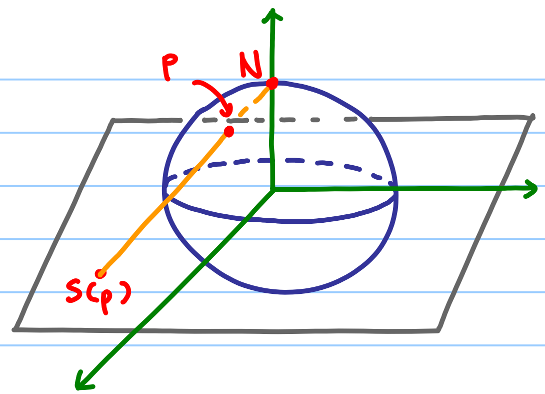
$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "s"'] & D \arrow[d, "s"] \\ R \arrow[r, "f"'] & R \end{tikzcd}$$

e.g. The stereographic projection $S : S^2 \setminus \{N\} \to \mathbb{C}$ (or $S : S^2 \to \hat{\mathbb{C}}$) is a covering transformation.

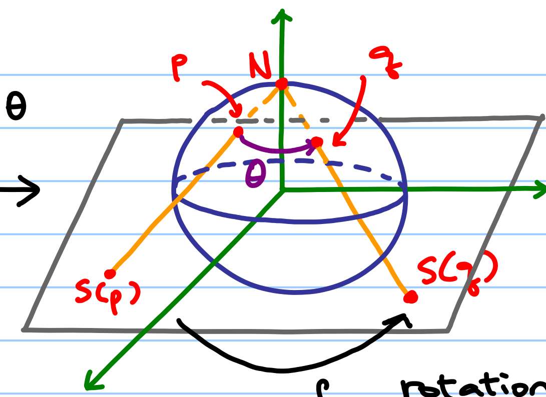
<u>Rmk</u> A covering transformation may not be invertible since it is not necessarily injective.

e.g. $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ is a covering transformation which is not invertible.

e.g.



g = rotation of angle θ about the z-axis



f = rotation of θ about 0 in $\mathbb{C}$

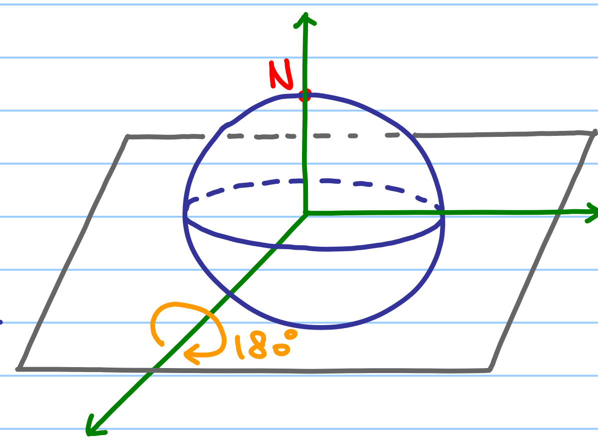
$$\Longrightarrow \quad \begin{tikzcd} S^2 \arrow[r, "g"] \arrow[d, "S"'] & S^2 \arrow[d, "S"] \\ \hat{\mathbb{C}} \arrow[r, "f"'] & \hat{\mathbb{C}} \end{tikzcd} \quad \text{commutes.}$$

So g is a lift of f (with respect to S)

e.g. Let $\tilde{T}: S^2 \to S^2$ be the rotation of 180° about the x-axis

i.e. $\tilde{T}(a,b,c) = (a,-b,-c)$.

Then $\tilde{T}$ is a lift of the inversion with respect to $S: S^2 \to \hat{\mathbb{C}}$.



Pf: To see this, let $z = \dfrac{a+ib}{1-c} = S(a,b,c)$.

Then $T(z) = \dfrac{1}{z} = \dfrac{1-c}{a+ib}$

$$= \frac{(1-c)(a-ib)}{a^2+b^2}$$

$$= \frac{(1-c)(a-ib)}{1-c^2} = \frac{a-ib}{1+c} = S(a,-b,-c).$$

$\Rightarrow T(S(a,b,c)) = S(a,-b,-c)$

i.e. $T(S(a,b,c)) = S(\tilde{T}(a,b,c)) \quad \forall (a,b,c) \in S^2 \setminus \{N, \overset{\overset{(0,0,-1)}{\parallel}}{S}\}$

$$\Longrightarrow \quad \begin{tikzcd} S^2 \arrow[r, "\tilde{T}"] \arrow[d, "S"'] \arrow[dr, phantom, "\circlearrowright"] & S^2 \arrow[d, "S"] \\ \hat{\mathbb{C}} \arrow[r, "T"'] & \hat{\mathbb{C}} \end{tikzcd}$$

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