THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 9 10 practice problems
26th March 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. Recall that the hyperbolic tangent function is defined as

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

   (a) Show that $\tanh(x)$ is strictly increasing, i.e., $\tanh'(x) > 0$ for any x.

   (b) Hence we can take the inverse function of $\tanh$, show that for $-1 < x < 1$, we have $\tanh^{-1}(y) = \frac{1}{2}\ln\frac{1+y}{1-y}$.

   (c) Prove also that $\tanh(x)$ is subadditive, i.e. $\tanh(x+y) \leq \tanh(x) + \tanh(y)$ whenever $x, y \geq 0$. (Hint: consider for fixed $y > 0$, and $f(x) = \tanh(x+y) - \tanh(x) - \tanh(y)$.)

   (d) Usually, we define distance (or known as a metric) to be any function d satisfying $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$ and the triangle inequality. Now we define $p(z,w) = \frac{|z-w|}{|1-z\bar{w}|}$, explain why this is still a distance function on $\mathbb{D}$. This is known as the pseudo-hyperbolic distance. (Hint: use the previous results)

   (e) Using the known facts about hyperbolic distance d, show that $p(f(z), f(w)) = p(z,w)$ for any Mobius transformation $f \in H$. (Remark: Proving this from scratch requires a famous result in complex analysis called the Schwarz lemma.)

   (f) The reason why we don't use this "nicer-looking" distance function is because it fails to satisfy additivity along a hyperbolic straight line. As an exercise, show that $p(x,0) + p(0,y) \neq p(x,y)$ for $x < 0 < y$.

2. A hyperbolic circle is as we defined it in lecture 8 is just a circle inside $\mathbb{D}$ that does not intersect with ∂D. Recall that classically we just define a circle to be the set of points that is of fixed distance to a fixed center. We can do the same for hyperbolic geometry: we define a hyperbolic circle to be $C(z_0, r) = \{z \in \mathbb{D} : d(z, z_0) = r\}$ where $r > 0, z_0 \in \mathbb{D}$ and d is the hyperbolic distance.

   (a) Prove that $C(z_0, r)$ is actually just a Euclidean circle, so the two definitions of hyperbolic circle coincide.

   (b) By part (a), we have $C(z_0, r) = C_{\text{Euclidean}}(z_0', r')$, where the latter denotes the usual circle of radius r' centered at z_0', try to find the relationship between z_0, z_0' and r, r'.

(c) Show that the hyperbolic length of $C(z_0, r)$ is $2\pi \sinh r$.

3. Determine the hyperbolic distance between $\frac{1}{2}$ and $\frac{1}{4} + \frac{i}{2}$.

4. (Sanity check) Let's check that the hyperbolic straight line is really shorter than classical straight line in the following example: Say $p = \frac{i}{2}$, $q = \frac{1}{2}(1+i)$, let $\gamma(t) = (1-t)p + tq$ for $t \in [0,1]$ be a parametrization of straight line from p to q. Compare $d(p,q)$ and $\ell(\gamma)$ the hyperbolic length of γ.

5. We have seen that any Mobius transformation $f \in H$ is an isometry of $\mathbb{D}$ under hyperbolic distance, i.e. $d(f(z), f(w)) = d(z,w)$ for any z, w. The converse is also (almost) true. Prove that any (orientation-preserving) isometry of $(\mathbb{D}, d)$ is a Mobius transformation in the hyperbolic group H. (You can ignore the orientation-preserving part for now, basically it means that any isometry is either an $f \in H$ or that its conjugate $\bar{f} \in H$.)