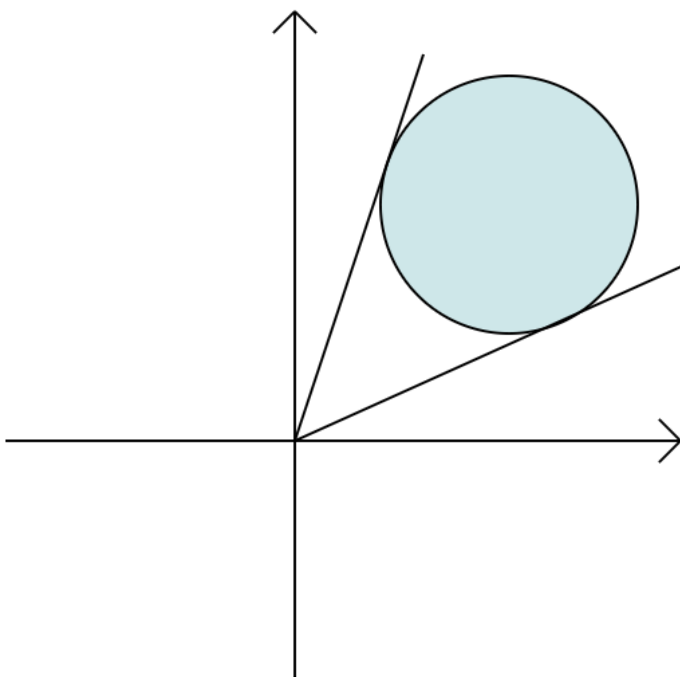
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 8 practice problems solution
16th March 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. Pick $p, q \in \mathbb{D}$, it is not easy to tell what would be a horocycle through these points in the disk picture but at least we know it is a Steiner circle of 1st kind. What we can do is to follow our analysis of Steiner circles, and take a transformation that map $p \mapsto 0$ and $q \mapsto \infty$. In this new picture, the original unit circle would be mapped to some circle in the plane (it is impossible to be a straight line because the unit circle doesn't pass through q). All the Steiner circles of 1st kind are the straight line through the origin. Being a horocycle just means that we want the straight line to be tangent to the circle, this is obviously possible by drawing a picture. And there would be exactly two lines tangent to the circle, corresponding to two horocycle.



2. There was a mistake in the original question, see this solution for the corrected answer.

   Let's rotate the figure in lecture 8 page 3 so that the line segment $\overline{ps}$ is on the positive x-axis, where p is the origin of the figure. We can write down the coordinates of the points for concrete computation. Let $x = |\overline{pr}|$, we would like to describe the hyperbolic straight line $\overline{srq}$ more explicitly. Recall that it is a circular arc, so we just have to determine the circle's radius and centre. Note that $r = xe^{i\theta}$ and $r^* = \frac{1}{x}e^{i\theta}$, which lies on the circle as well by a lemma in lecture 7. By flipping the figure along the line rr^*, we see that it is fixed in the figure, which means that rr^* is a diameter of the circle. Therefore the radius can be expressed as $\frac{1}{2}(\frac{1}{x} - x)$ and the center is at $c = \frac{1}{2}(\frac{1}{x} + x)e^{i\theta}$. The equation of circle can be expressed as $|z - c| = \frac{1}{2}(\frac{1}{x} - x)$. We can use this to relate to the point $q = e^{2i\theta}$ because it is the intersection with the unit circle $|z|^2 = z\bar{z} = 1$.

Now we can multiply by 2 and expand $|2z - 2c|^2 = (\frac{1}{x} - x)^2 = \frac{1}{x^2} - 2 + x^2$. The LHS simplifies to

$$\begin{aligned} |2z - 2c|^2 &= 4(z - c)(\bar{z} - \bar{c}) \\ &= 4[z\bar{z} - (\bar{z}c + z\bar{c}) + c\bar{c}] \\ &= 4 + \left(\frac{1}{x} + x\right)^2 - 4(\bar{z}c + \bar{c}z) \end{aligned}$$

The $x^2 + \frac{1}{x^2}$ on both sides cancel out and we are left with $\bar{z}c + \bar{c}z = 2$. Now we can substitute $q = e^{2i\theta}$ into the equation and obtain $\bar{q}c + \bar{c}q = \frac{1}{2}((x + \frac{1}{x})e^{i\theta}e^{-2i\theta} + (x + \frac{1}{x})e^{-i\theta}e^{2i\theta}) = 2$. Writing $e^{i\theta} = \cos\theta$, we get $(x + \frac{1}{x})\cos\theta = 2$. So

$$\cos\theta = \frac{2}{x + \frac{1}{x}} = \frac{2x}{x^2 + 1}$$

This is really just a long way of saying that the circles always have their centers lying on the line $\mathrm{Re}(c) = 1$.