Solution 5

Hand in Ex 3.2, no. 3, Supp. Ex no. 4, 7 by March 7.

Ex 3.2

1. Show that $\left\{(2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.

   Solution. Let $\psi_n(x) = (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)$. Then, for $m, n \geq 1$,

$$\begin{aligned}\langle \psi_n, \psi_m\rangle &= \int_0^l (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)(2/l)^{1/2}\sin(m-\frac{1}{2})(\pi x/l)dx \\ &= \frac{2}{l}\int_0^l \sin(n-\frac{1}{2})(\pi x/l)\sin(m-\frac{1}{2})(\pi x/l)dx \\ &= \frac{1}{l}\int_0^l (\cos(n-m)(\pi x/l) - \cos(n+m-1)(\pi x/l))\,dx \\ &= \frac{1}{l}\int_0^l \cos(n-m)(\pi x/l)dx \\ &= \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

   since $n + m \neq 1$. Hence $\left\{(2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.

3. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal on $[-1,1]$, and find constants a and b so that $f_2(x) = x^2 + ax + b$ are orthogonal to both f_0 and f_1 on $[-1,1]$. What are the normalizations of f_0, f_1 and f_2?

   Solution. f_0 and f_1 are orthogonal since

$$\langle f_0, f_1\rangle = \int_{-1}^1 1\cdot x dx = \left.\frac{x^2}{2}\right|_{-1}^1 = 0.$$

   If f_2 is orthogonal to both f_0 and f_1, then

$$0 = \langle f_0, f_2\rangle = \int_{-1}^1 (x^2+ax+b)dx = \left[\frac{x^3}{3} + \frac{ax^2}{2} + bx\right]_{-1}^1 = \frac{2}{3} + 2b,$$

   and

$$0 = \langle f_1, f_2\rangle = \int_{-1}^1 x(x^2+ax+b)dx = \left[\frac{x^4}{4} + \frac{ax^3}{3} + \frac{bx^2}{2}\right]_{-1}^1 = \frac{2a}{3}.$$

   Hence $a = 0$ and $b = -\frac{1}{3}$. Since

$$\|f_0\| = \sqrt{2}, \quad \|f_1\| = \sqrt{\frac{2}{3}}, \quad \|f_2\| = \sqrt{\frac{8}{45}},$$

   their normalizations are given by

$$\frac{1}{\sqrt{2}}, \quad \sqrt{\frac{3}{2}}x, \quad \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right).$$

4. Suppose $\{\phi_n\}$ is an orthonormal set in $PC(0,l)$, and let ϕ_n^+ and ϕ_n^- be the even and odd extension of ϕ to $[-l,l]$. Show that $\{2^{-1/2}\phi_n^+\} \cup \{2^{-1/2}\phi_n^-\}$ is an orthonormal set in $PC(-l,l)$.

   Solution. Note that

$$\begin{aligned}
\langle \phi_n^+, \phi_m^+ \rangle &= \int_{-l}^{l} \phi_n^+(x)\phi_m^+(x)dx \\
&= \int_0^l \phi_n^+(x)\phi_m^+(x)dx + \int_{-l}^0 \phi_n^+(x)\phi_m^+(x)dx \\
&= \int_0^l \phi_n(x)\phi_m(x)dx + \int_0^l \phi_n(x)\phi_m(x)dx \\
&= \begin{cases} 2 & \text{if } n=m, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

   Hence $\{2^{-1/2}\phi_n^+\}$ is an orthonormal set in $PC(-l,l)$. Similarly, one can show that $\{2^{-1/2}\phi_n^-\}$ is an orthonormal set in $PC(-l,l)$. It remains to show that $\langle \phi_n^+, \phi_m^- \rangle = 0$ for all m,n. Indeed,

$$\begin{aligned}
\langle \phi_n^+, \phi_m^- \rangle &= \int_{-l}^{l} \phi_n^+(x)\phi_m^-(x)dx \\
&= \int_0^l \phi_n^+(x)\phi_m^-(x)dx + \int_{-l}^0 \phi_n^+(x)\phi_m^-(x)dx \\
&= \int_0^l \phi_n^+(x)\phi_m^-(x)dx + \int_l^0 \phi_n^+(-x)\phi_m^-(-x)(-dx) \\
&= \int_0^l \phi_n(x)\phi_m(x)dx + \int_0^l \phi_n(x)(-\phi_m(x))dx \\
&= 0.
\end{aligned}$$

Supplementary Exercise. The space $R[a,b]$ consisting of all Riemann integrable functions is endowed with the inner product

$$\langle f,g \rangle = \int_a^b f(x)g(x)dx.$$

Set $\|f\| = \sqrt{\langle f,f \rangle}$.

1. Let $\{v_j\}_{j=1}^N$, $N \le \infty$, be an orthonormal set in V. Prove Bessel's inequality

$$\sum_{j=1}^{N} \langle v, v_j \rangle^2 \le \|v\|^2.$$

Solution. First assume that $N < \infty$. By expanding the inner product, we have

$$\begin{aligned}
0 \le \left\| v - \sum_{j=1}^{N} \langle v, v_j \rangle v_j \right\|^2 &= \left\langle v - \sum_{j=1}^{N} \langle v, v_j \rangle v_j, v - \sum_{j=1}^{N} \langle v, v_j \rangle v_j \right\rangle \\
&= \langle v, v \rangle - \sum_{j=1}^{N} \overline{\langle v, v_j \rangle} \langle v, v_j \rangle - \sum_{j=1}^{N} \langle v, v_j \rangle \langle v_j, v \rangle + \sum_{j=1}^{N} \langle v, v_j \rangle \overline{\langle v, v_j \rangle} \\
&= \|v\| - \sum_{j=1}^{N} |\langle v, v_j \rangle|^2.
\end{aligned}$$

Rearranging yields the inequality

$$\sum_{j=1}^{N} \langle v, v_j \rangle^2 \le \|v\|^2.$$

The same inequality holds when $N = \infty$ since each summand is non-negative.

2. Establish Cauchy-Schwarz inequality $|\langle u, v \rangle| \le \|u\| \|v\|$ in an inner product space and then use it to prove the triangle inequality

$$\|u + v\| \le \|u\| + \|v\|.$$

Do it in both real and complex cases.

Solution. By expanding the inner product, we have

$$\begin{aligned}
0 \le \left\| u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v \right\|^2 &= \left\langle u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v, u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v \right\rangle \\
&= \langle u, u \rangle - \frac{\overline{\langle u, v \rangle}}{\langle v, v \rangle} \langle u, v \rangle - \frac{\langle u, v \rangle}{\langle v, v \rangle} \langle v, u \rangle + \frac{\langle u, v \rangle}{\langle v, v \rangle} \frac{\overline{\langle u, v \rangle}}{\langle v, v \rangle} \langle v, v \rangle \\
&= \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2} - \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \frac{|\langle u, v \rangle|^2}{\|v\|^2} \\
&= \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2}.
\end{aligned}$$

Rearranging yields the Cauchy-Schwarz inequality.

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v \rangle \\
&= \langle u, u \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} + \langle v, v \rangle \\
&= \|u\|^2 + 2\mathrm{Re}\langle u, v \rangle + \|v\|^2 \\
&\le \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2 \\
&\le \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

Taking square root on both sides yields the triangle inequality.

3. Show that for $f \in R[a, b]$,

$$\|f\| \le \sqrt{b - a} \|f\|_\infty,$$

where $\|f\|_\infty = \sup_x |f(x)|$ is the sup-norm of f. Then use it to show f_n tends to f in L^2-norm if f_n tends to f uniformly, but the converse is not true.

Solution.

$$\|f\|^2 = \int_a^b |f(x)|^2 dx \le \int_a^b \|f\|_\infty^2 dx = (b-a)\|f\|_\infty^2.$$

Taking square root on both sides we have

$$\|f\| \le \sqrt{b-a}\|f\|_\infty.$$

It is clear that f_n tends to f in L^2-norm if f_n tends to f uniformly since

$$\|f_n - f\| \le \sqrt{b-a}\|f_n - f\|_\infty$$

To see that the converse is not true, consider, on $[0,1]$, the sequence

$$f_n(x) = \begin{cases} n & x = 0 \\ \text{linear} & 0 < x < \frac{1}{n^2} \\ 0 & \frac{1}{n^2} \le x \le 1. \end{cases}$$

Then $\|f_n\| = \dfrac{1}{\sqrt{2n}}$, so that f_n converges to 0 in L^2-norm. On the other hand, $\|f_n\|_\infty = n$, hence f_n does not converge uniformly (as it is not bounded in sup-norm).

4. The sequence $\{f_n\}$ is called pointwisely convergent to f if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in [a,b]$. Construct (a) a pointwisely convergent but not L^2-convergent sequence, and (b) an L^2-norm convergent but not pointwisely convergent sequence. You may work on $[0,1]$.

Solution.

(a) Consider the sequence

$$f_n(x) = \begin{cases} 2n^2 x & 0 \le x \le \frac{1}{2n}, \\ 2n(1-nx) & \frac{1}{2n} < x \le \frac{1}{n}, \\ 0 & \frac{1}{n} < x \le 1. \end{cases}$$

Then $f_n(x)$ converges to 0 for every $x \in [0,1]$. However,

$$\begin{aligned} \|f_n\|^2 &= \int_0^{\frac{1}{2n}} (2n^2x)^2 dx + \int_{\frac{1}{2n}}^{\frac{1}{n}} (2n(1-nx))^2 dx \\ &= 2\int_0^{\frac{1}{2n}} 4n^4x^2 dx \\ &= \frac{n}{3}. \end{aligned}$$

Hence $\{f_n\}$ is pointwisely convergent to 0 but not L^2-convergent (as it is unbounded in L^2-norm).

(b) Consider the sequence f_n defined in Question 3. Then $\{f_n\}$ is L^2-convergent to 0 but not pointwisely convergent (since $f_n(0) \to \infty$).

5. Let W be a subspace in V and $\{w_1, \ldots, w_n\}$ be an orthonormal basis of W. Suppose that $w_1 \in W$ satisfies $\langle u - w_1, w \rangle = 0$ for all $w \in W$. Show that w_1 is the orthogonal projection of u on W.

   Solution. Let $w \in W$. Since $w_1 - w \in W$, we have $\langle u - w_1, w_1 - w \rangle = 0$, and hence,

$$\begin{aligned}
\|u - w\|^2 &= \langle (u - w_1) + (w_1 - w), (u - w_1) + (w_1 - w) \rangle \\
&= \|u - w_1\|^2 + \langle u - w_1, w_1 - w \rangle + \langle w_1 - w, u - w_1 \rangle + \|w_1 - w\|^2 \\
&= \|u - w_1\|^2 + \|w_1 - w\|^2 \\
&\geq \|u - w_1\|^2.
\end{aligned}$$

   Thus

$$\|u - w_1\| \leq \|u - w\| \qquad \text{for all } w \in W.$$

   By Theorem 4.1, w_1 is the orthogonal projection of u on W.

6. Verify that the orthogonal projection of f on the subspace E_n (see page 3, Note 4) is equal to $S_n f$, the n-th partial sum of the Fourier series of f.

   Solution.

   Recall that

$$E_n = \left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos jx, \frac{1}{\sqrt{\pi}} \sin jx \right\rangle_{j=1}^{n}.$$

   By Theorem 4.1, the orthogonal projection of f on E_n is given by

$$\begin{aligned}
w^* &= \langle f, \frac{1}{\sqrt{2\pi}} \rangle \frac{1}{\sqrt{2\pi}} + \sum_{j=1}^{n} \langle f, \frac{1}{\sqrt{\pi}} \cos jx \rangle \frac{1}{\sqrt{\pi}} \cos jx + \sum_{j=1}^{n} \langle f, \frac{1}{\sqrt{\pi}} \sin jx \rangle \frac{1}{\sqrt{\pi}} \sin jx \\
&= \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2\pi}} dx \frac{1}{\sqrt{2\pi}} + \sum_{j=1}^{n} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \cos jx dx \frac{1}{\sqrt{\pi}} \cos jx \\
&\quad + \sum_{j=1}^{n} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \sin jx dx \frac{1}{\sqrt{\pi}} \sin jx \\
&= \frac{a_0}{2} + \sum_{j=1}^{n} (a_n \cos jx + b_n \sin jx) \\
&= S_n f.
\end{aligned}$$

7. Establish the identity

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

   by looking at the Parseval's identity for the function $f(x) = x$.

   Solution. Recall that $f(x) = x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$. By Parseval's identity,

$$\begin{aligned}
\int_{-\pi}^{\pi} x^2 dx &= \frac{\pi}{2} 0^2 + \pi \sum_{n=1}^{\infty} \left[0^2 + \left(\frac{2(-1)^{n+1}}{n} \right)^2 \right] \\
\frac{2\pi^3}{3} &= 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

   Rearranging the terms yields $\dfrac{\pi^2}{6} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.